Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

December 2020

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. Let Y be a connected space. Let $f\colon X \to Y$ be a quotient map, such that $f^{-1}(y)$ is connected for each $y \in Y$. Prove that X is connected.

2. Let $X = [0, 2)$ and let $Y = [0, 1) \cup (1, 2]$. Prove that X and Y are not homeomorphic. Are the one-point compactifications $\widehat{X}$ and $\widehat{Y}$ homeomorphic? Carefully justify.

3. A continuous map between Hausdorff spaces is *closed* if the image of each closed set is closed. A continuous map is *proper* if the inverse image of each compact subset is compact. Let X, Y be metric spaces.

   (a) Show that every continous proper map $f\colon X \to Y$ is closed.

   (b) Give an example of a map $g\colon X \to Y$ which is not closed. Justify.

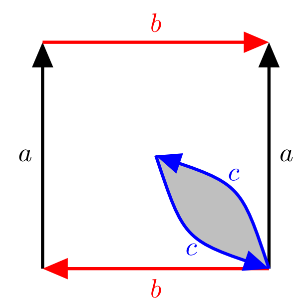
4. Let D be a dense subset of X, let Y be Hausdorff, and let $f, g\colon X \to Y$ be maps.
   Show that if $f|_D = g|_D$, then $f = g$.

Part II

5. Let X be obtained from the Klein bottle K by removing a small open disk, and identifying the antipodal points of the resulting boundary circle on K (see figure).

   (a) Use van Kampen's Theorem to find a presentation for $\pi_1(X)$.

   (b) Compute the homology $H_*(X)$ using a Δ-complex structure.



6. Let z_1 and z_2 be two distinct points in $\mathbb{RP}^2$. Let X be the space obtained by identifying z_1 with z_2. Describe a Δ–complex structure on X, and use it to compute $H_*(X)$.

7. (a) Show that if $\pi_1(X)$ is abelian and $A \subset X$ is a retract, then $\pi_1(A)$ is also abelian.

   (b) Show that the figure-8 graph is not a retract of the torus.

8. Let $X = T^2 - D^2$ be the torus with a disc removed. Determine all connected surfaces that are 3–fold covers of X. (You do not need to construct all covering spaces up to covering space isomorphism, just decide which surfaces arise as covers, up to homeomorphism.)

9. Explicitly construct all connected three-fold covers of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ up to covering space isomorphism.

Part III

10. Let K denote the Klein bottle, which contains an embedded Mobius band M. Let X be the space obtained by gluing two copies of K together along M. Compute $H_*(X)$ using the Mayer-Vietoris Theorem.

11. Let M be a connected n–manifold. Let D be an embedded closed n–disc in M. Show that if $\partial D \hookrightarrow M \setminus D^\circ$ is null-homotopic, then M is orientable.

12. Let T be the torus. Let $f\colon T \to T$, and let f_*, f^* be the induced maps on H_1 and H^1. Let F be the matrix representing f_*. Show that $\det F = \deg(f)$.
    Hint: Compute $f^*(xy)$, where x and y generate $H^1(T)$.

13. (a) Compute the cohomology ring $H^*(\mathbb{CP}^2; \mathbb{Z})$, using Poincaré duality or another method.
    (b) Show that there is no degree one map from S^4 to $\mathbb{CP}^2$.

14. Compute the following:
    (a) $H_*(\mathbb{R}P^2 \times \mathbb{R}P^3; G)$, for both $G = \mathbb{Z}$ and $G = \mathbb{Z}/2$.
    (b) $H_*(\mathbb{R}P^n; G)$ where $G = \mathbb{Q}/\mathbb{Z}$.