Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Fall 2020

Instructions: Do 7 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 7 problems, identify which 7 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. Show that if X is Hausdorff and locally compact then the one-point compactification of X is Hausdorff.

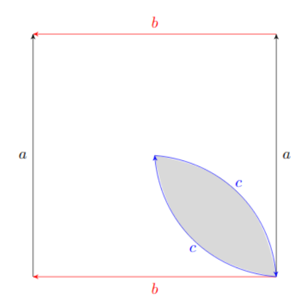
2. Prove that if $\{X_i \,|\, i \in \mathcal{I}\}$ is a family of connected spaces such that $\cap_{i\in\mathcal{I}} X_i \neq \emptyset$, then $\cup_{i\in\mathcal{I}} X_i$ is connected.

3. Let $X = \mathbb{R}$ with the basis for its topology all open intervals (a, b) for $a < b$ in $\mathbb{R}$.
   Let $Y = \mathbb{R}$ with the basis for its topology all closed-open intervals $[a, b)$ for $a < b$ in $\mathbb{R}$.
   Let $f\colon X \to Y$ be given by $x \mapsto x$. Let $g\colon Y \to X$ be given by $x \mapsto x$.
   Justify for each of f and g, whether these functions are continuous, open, and/or closed.
   (Here, *closed* means the image of each closed set is closed, and likewise for *open*).

4. (a) Let X be a Hausdorff topological space. If $\{x_n\}$ is a convergent sequence in X, prove that $\lim_{n\to\infty} x_n$ is unique.
   (b) Suppose $f\colon X \to Y$ is a continuous surjective function. Show that if X is compact and Y is Hausdorff, then f is a quotient map.

Part II

5. Let X be obtained from the torus T^2 by removing a small open disk, and identifying the antipodal points of the resulting boundary circle on T^2 (see figure).



   (a) Use van Kampen's Theorem to write down a presentation for $\pi_1(X)$.
   (b) Compute the homology $H_*(X)$ using a Δ-complex structure. Verify that your answer agrees with part (a).

6. For $n \geq 2$, let X_n be the quotient of n 2-disks $\{D_1^2, \ldots, D_n^2\}$ with their boundary circles identified. Let $Y_n = S^1 \sqcup_f D^2$, where $f(z) = z^n$.
   (a) Prove that $\pi_1(Y_n) = \mathbb{Z}/n\mathbb{Z}$, and that X_n is the universal covering space for Y_n.
   (b) Use X_6 to describe all isomorphism classes of path-connected covering spaces of Y_6.

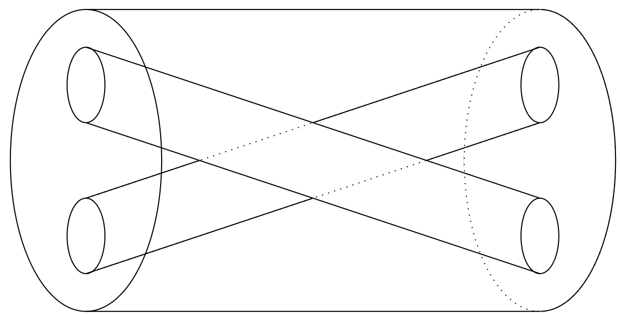
7. Let X be a CW-complex such that $H_1(X) = \mathbb{Z}/3$. Let T^3 be the 3-torus. Prove that every continuous map $f\colon X \to T^3$ is homotopic to a constant map.

8. Describe three connected non-homeomorphic 2–fold covering spaces of $\mathbb{R}P^2 \vee S^1$.
   (a) Justify algebraically.
   (b) Sketch the covers.

Part III

9. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology groups in all dimensions. Are they homotopy equivalent? Justify.

10. Let X be the space consisting of a solid torus, with an open neighbourhood of a curve running twice around it's interior removed, as illustrated below (glue the left end to the right end by the identity map).



Use the Mayer-Vietoris Theorem to compute the homology groups of X.

11. (a) Compute the reduced homology group $\tilde{H}_n(\mathbb{R}P^n)$ for all n. Justify.

    (b) Use the long exact sequence of a pair to compute $\tilde{H}_{n-1}(\mathbb{R}P^n)$ for all n. Justify.

12. Let $m,\ n \geq 1$.

    (a) Describe the cohomology rings $H^*(\mathbb{R}P^m \vee \mathbb{R}P^n; \mathbb{Z}/2)$ and $H^*(\mathbb{R}P^m \times \mathbb{R}P^n; \mathbb{Z}/2)$.

    (b) Show that $\mathbb{R}P^m \vee \mathbb{R}P^n$ cannot be a retract of $\mathbb{R}P^m \times \mathbb{R}P^n$.

13. Let T denote the torus and K denote the Klein bottle.

    (a) Prove that for any map $f\colon T \to K$, the map $f^*\colon H^2(K;\mathbb{Z}_2) \to H^2(T;\mathbb{Z}_2)$ is trivial.

    (b) Using the cup product on $H^*(T)$, show that for any non-zero $\alpha \in H^1(T)$, there exists $\beta \in H^1(T)$ such that $\alpha \cup \beta \neq 0$.