

Topology Qualifying Exam

Mathematics Program, CUNY Graduate Center

Spring 2015

Instructions: Do at least 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. Please justify your answers and include statements of any theorems you cite.

Part I

1. Define a family of functions in $\text{Hom}([0, 1], \mathbb{R})$ by $\mathcal{F} = \{f_a : 0 < a \leq 1\}$ where $f_a(x) = 1 - \frac{x}{a}$. Prove or disprove: \mathcal{F} is compact (in the compact-open topology).
2. A map between Hausdorff spaces is *closed* if the image of each closed set is closed. A map is *proper* if the inverse image of each compact subset is compact.
 - (a) Let X, Y be metric spaces. Show that every proper map $f: X \rightarrow Y$ is closed.
 - (b) Give an example of a map $g: X \rightarrow Y$ which is not closed. Justify.
3.
 - (a) Show that a compact Hausdorff space is closed.
 - (b) Show that if X, Y are connected then $X \times Y$ is connected.

Part II

4. Let X be the CW complex obtained by attaching a 1-cell to a 0-cell to obtain a circle, and then attaching a 2-cell to the circle by the attaching map $z \mapsto z^4$.
 - (a) Compute $\pi_1(X)$.
 - (b) Prove or disprove: every map $X \rightarrow S^1$ is null-homotopic.
 - (c) Prove or disprove: every map $X \rightarrow \mathbb{R}P^2$ is null-homotopic.
5. Let $T = S^1 \times S^1$ be the torus.
 - (a) Show that every finite covering space of T is homeomorphic to T .
 - (b) Find two four-fold covering maps $f: T \rightarrow T$ and $g: T \rightarrow T$ such that there do NOT exist homeomorphisms $\alpha: T \rightarrow T$ and $\beta: T \rightarrow T$ such that $\alpha \circ g = f \circ \beta$.
6. Let X_i be a circle with basepoint x_i , and let α_i be a generator of $\pi_1(X_i)$. Let Y be $X_1 \vee X_2$ with a Möbius band attached along the path $\alpha_1 \cdot \alpha_2$. Use the van Kampen Theorem to write down a presentation for $\pi_1(Y)$.
7. Let X be the subset of \mathbb{R}^3 obtained by rotating two tangent circles in the plane around an axis in the same plane as shown in the figure.



- (a) Compute $\pi_1(X)$.
- (b) Describe a non-trivial covering space of X .

8. Determine using the fundamental group if the spaces

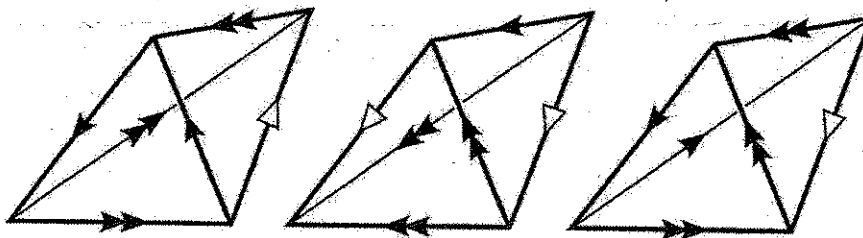
$$X = \mathbb{R}^3 - (\{(0, 0, z) | z \in \mathbb{R}\} \cup \{(x, y, 0) | x^2 + y^2 = 1\}) \quad \&$$

$$Y = \mathbb{R}^3 - (\{(0, 0, z) | z \in \mathbb{R}\} \cup \{(x, y, 0) | (x - 2)^2 + y^2 = 1\})$$

are homotopic.

Part III

9. For a closed surface M with an embedded disc D , let $M^+ = (M - \dot{D}) \cup_{\partial} (M - \dot{D})$ by an orientation-preserving gluing, and let $M^- = (M - \dot{D}) \cup_{\partial} (M - \dot{D})$ by an orientation-reversing gluing. Determine (a) M^+ for the sphere, (b) M^- for the torus, (c) M^+ for the Klein bottle, (d) M^- for the projective plane. Justify.
10. Let X_i be a circle with basepoint x_i , and let α_i be a generator of $\pi_1(X_i)$. Let Y be $X_1 \vee X_2$ with a Möbius band attached along the path $\alpha_1 \cdot \alpha_2$. Write down a cell structure for Y and use it to compute the homology groups of Y .
11. Let X be the CW-complex obtained by identifying faces of the three 3-simplices as indicated in the figure below. (A unique simplicial identification preserves the arrows as shown.)



- (a) Show that X has one 0-cell, three 1-cells, six 2-cells, and three 3-cells.
- (b) Give the cellular chain complex for X .
- (c) The homology groups of X are given by

$$H_n(X) = \begin{cases} \mathbb{Z} & n=0, 2, 3 \\ \mathbb{Z}/3 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

Verify this fact for $n = 1$ and $n = 3$.

- (d) Does X have the homotopy type of a closed 3-manifold? Why or why not? (You may use the homology groups given in (c), even the ones you did not compute.)
12. Let $m, n \geq 1$.
- (a) Describe the cohomology rings $H^*(\mathbb{R}P^m \wedge \mathbb{R}P^n; \mathbb{Z}/2)$ and $H^*(\mathbb{R}P^m \times \mathbb{R}P^n; \mathbb{Z}/2)$.
- (b) Show that $\mathbb{R}P^m \wedge \mathbb{R}P^n$ cannot be a retract of $\mathbb{R}P^m \times \mathbb{R}P^n$.
13. The suspension SX of X is the quotient space

$$SX = (X \times [0, 1]) / \{(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X\}.$$

Prove that $\tilde{H}_{n+1}(SX) \cong \tilde{H}_n(X)$.

14. (a) Let T denote the torus and K denote the Klein bottle. Prove that for any map $f : T \rightarrow K$ the map $f^* : H^2(K; \mathbb{Z}_2) \rightarrow H^2(T; \mathbb{Z}_2)$ is trivial.
- (b) Using the cup product structure on the torus T that for any non-zero $\alpha \in H^1(T)$ there exists $\beta \in H^1(T)$ such that $\alpha \cup \beta \neq 0$.