Topology Qualifying Exam

CUNY Graduate Center Mathematics Program

Spring 2016

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers, and include the names or the precise statements of any theorems you cite.

Part I

1. Call a space KC if it has the property that every compact set is closed.
   (a) Prove that every Hausdorff space X is KC.
   (b) Give an example of a space Y that is KC but not Hausdorff. Justify.

2. Let X be a contractible space and let Y be any topological space. Show that $X \times Y$ is homotopy equivalent to Y.

3. Prove that the following two descriptions of $\mathbb{RP}^2$ are homeomorphic:
   Let $X = S^2/x \sim (-x)$, where x and $-x$ are antipodal points on S^2.
   Let $Y = D^2/x \sim (-x)$, where x and $-x$ are antipodal points on ∂D^2.

4. Prove or disprove:
   (a) There is a quotient map $f\colon (0,1) \to [0,1]$.
   (b) There is a quotient map $g\colon [0,1] \to (0,1)$.
   (c) The one-point compactification of $\mathbb{R}$ is homeomorphic to $\mathbb{RP}^1$.
   (d) The one-point compactification of $\mathbb{R}^2$ is homeomorphic to $\mathbb{RP}^2$.

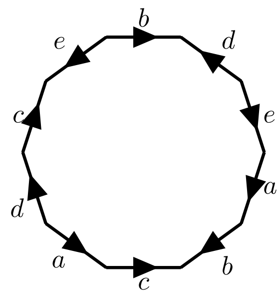
Part II

5. Let X be the space formed by identifying the boundaries of two Möbius bands using the degree two map $S^1 \to S^1$. Use van Kampen's theorem to write down a presentation for $\pi_1(X)$.

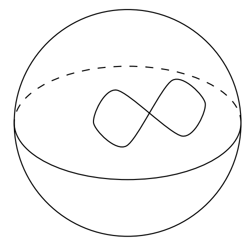
6. Explicitly enumerate all 3-fold covers of the Klein bottle.

7. Let S be the closed orientable surface of genus 2. Let X be the regular (or normal) cover of S with deck transformation group $\mathbb{Z} \times \mathbb{Z}$.
   (a) Describe X explicitly as a subspace of $\mathbb{R}^3$.
   (b) For this covering map, describe the lifts of a standard generating set of $\pi_1(S)$ and of the commutator subroup $[\pi_1(S), \pi_1(S)]$.

8. Let F_n be the free group with n generators. Show that for any $k < \infty$, the free group F_n has only a finite number of subgroups of index k.



(a) Figure for Q10



(b) Figure for Q11

Figure 1

9. Give an example of a group that cannot be the fundamental group of a one or two dimensional compact manifold, possibly with boundary, and justify carefully why this is the case. Then, construct a space with that group as its fundamental group.

10. State the classification of closed orientable surfaces. Compute the Euler characteristic of the surface obtained by identifying the sides of the polygon as indicated in Figure 1(a), and identify which surface it is.

Part III

11. Let X be the space obtained by gluing the boundary of a disc to S^2 along a figure-eight curve, as shown in Figure 1(b). Give an explicit cell structure for X, and use it to compute the homology groups of X.

12. (a) Show that a Möbius band retracts onto S^1.
    (b) Show that a Möbius band does not retract onto its boundary.

13. Let X be the space formed from $S^2 \times I$ by identifying $(x, 1)$ with $(f(x), 0)$, where f is an orientation reversing homeomorphism $f\colon S^2 \to S^2$. Use the Mayer-Vietoris sequence to compute the homology groups of X.

14. Write down an explicit Δ-complex structure on the Klein bottle, and use it to compute the cup product structure on cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

15. (a) Calculate the cup product ring structure on $H^*(\mathbb{CP}^2; \mathbb{Z})$, using Poincaré duality, or any other method.
    (b) Show that there is no degree one map from S^4 to $\mathbb{CP}^2$.

16. What can you say about the homology of a closed connected orientable 3-manifold M whose fundamental group is the free group on two generators?