CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
FALL 2009
3 hours

Instructions. Do any eight questions and no more. Partial credit will be awarded only on those eight questions. Please specify on the cover of your answer book the problems to be graded. Prove all answers, indicating clearly any well-known results you are using.

1. **a.** Show that there is no simple group of order 36.
b. Show that there is no simple group of order p^2q^2, for distinct primes p and q.

2. Let G be a group with subgroups H and K.
a. Show HK is a subgroup if and only if $HK = KH$.
b. Show $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$.

3. (*A proof of Cauchy's theorem*) Let G be a group whose order is divisible by a prime number p and let H be the cyclic subgroup of the symmetric group S_p generated by the p-cycle $(12\cdots p)$. Let

$$X = \{(a_1, \dots, a_p) \in G \times \cdots \times G : a_1a_2\cdots a_p = 1\}.$$

For $(a_1, \dots, a_p) \in X$ and $\sigma \in H$ define

$$(a_1, \dots, a_p) \circ \sigma = (a_{\sigma(1)}, \dots, a_{\sigma(p)}).$$

a. Show that $\circ$ defines an action of H on X.
b. Show $\#X = |G|^{p-1}$.
c. The orbit of $(1, \dots, 1) \in X$ has exactly one element. Show that there is another element of X whose orbit has exactly one element.
d. Conclude that there is an element of order p in G.

4. Let R be a commutative ring and let M, M', M'' be three R-modules. Suppose

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \qquad \text{(A)}$$

is exact. Let N be another R module.
a. Prove that the following sequence is exact:

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M', N).$$

b. Show by example for a suitable exact sequence of the form (A) and R-module N, the following need not be exact:

$$0 \to N \otimes_R M' \to N \otimes_R M \to N \otimes_R M'' \to 0.$$

Suggestion: You can give a simple example with $R = \mathbb{Z}$.

5. **a.** Prove that in a Euclidean domain, every ideal is principal.
b. In the integral domain $S = \mathbb{Z}[\sqrt{-13}]$, consider the ideal $I = 2S + (1 + \sqrt{-13})S$. Decide whether or not I is principal and justify your answer.
Suggestion. Use the norm $N(a + b\sqrt{-13}) = a^2 + 13b^2$, with $a, b \in \mathbb{Z}$.

(continued overleaf)

6. Let R be a ring with 1 such that $x^2 = x$ for all $x \in R$.
   a. Show that for all $x \in R, 2x = 0$.
   b. Show that R is commutative.
   c. Show that every finitely generated ideal of R is principal.

7. Let $\mathbb{F}_2$ be the field with two elements. For $f(x) = x^4 + x^3 + 1 \in \mathbb{F}_2[x]$, let $E = \mathbb{F}_2[x]/(f)$.
   a. Show E is a field and find $|E|$.
   b. How many generators for the cyclic group $E^\times$ are there? Find one (in coset notation).
   ~~c. Find the inverse of $x^3 + 1$ in E (in coset notation).~~
   d. Find a subfield of E with exactly 4 elements.
   e. Does E contain a subfield with exactly 8 elements?

8. Let K be a finite extension of $\mathbb{Q}$. Show that K does not contain infinitely many roots of unity.

9. **Theorem:** If E is a finite, separable extension of a field k then the extension must be simple (in other words $E = k(\alpha)$ for some $\alpha \in E$).
   Prove this theorem with your own proof or by filling in the details of the following steps:
   a. Show the theorem is true if k is finite.
   b. Assume k infinite and suppose $E = k(\alpha, \beta)$. Let $\sigma_1, \cdots, \sigma_n$ be the distinct embeddings of E in a closure k^a and consider

$$f(X) = \prod_{i \neq j} (\sigma_i \alpha + X\sigma_i \beta - \sigma_j \alpha - X\sigma_j \beta).$$

   Show that f is not identically 0 and there is some $c \in k$ with $f(c) \neq 0$.
   c. Show that $E = k(\alpha + c\beta)$.

10. Find the Galois groups of
   a. $x^5 - 1$ over $\mathbb{Q}$,
   b. $x^5 - 1$ over $\mathbb{F}_3$,
   c. $x^5 - 1$ over $\mathbb{F}_5$,
   d. $x^3 + 3x - 1$ over $\mathbb{Q}$,
   e. $x^4 - 7$ over $\mathbb{Q}(i)$,
   f. $x^4 - 7$ over $\mathbb{Q}$.
   (Recall that the discriminant of $x^3 + bx + c$ is $-4b^3 - 27c^2$.)

11. Let $E = k(\alpha)$ be a simple algebraic extension of k. Let $\sigma : k \hookrightarrow L$ be an embedding of k into the algebraically closed field L. Show how σ can be extended to an embedding of E into L.

12. Let K have characteristic p and consider $f(x) = x^p - x + a \in K[x]$. If α is a root of f in some extension of K, show that the extension $K(\alpha)$ of K is Galois and find its Galois group.