ALGEBRA QUALIFYING EXAM
SEPTEMBER, 2012

Instructions: Do eight problems. Do not do more than eight problems.

(1) Let k be a field.
   (a) Prove the Division Algorithm for the polynomial ring $k[x]$: If a and b are polynomials in $k[x]$ with b nonzero, then there exist unique polynomials $q, r \in k[x]$ such that
   $$a = qb + r$$
   and $r = 0$ or $\deg r < \deg g$.
   (b) Use part (a) to show that $k[x]$ is a principal ideal domain (PID).
   (c) Explain, with proof, whether $k[x, y]$ is a PID.

(2) (a) Prove that a group of order 160 is not simple.
   (b) Prove that a group of order pqr for distinct primes p, q, r is not simple.

(3) Let R be a ring and let M be a left R-module.
   (a) Prove that the following three conditions are equivalent:
      (i) M satisfies the ascending chain condition on submodules.
      (ii) Every nonempty set of submodules of M contains a maximal element under inclusion.
      (iii) Every submodule of M is finitely generated.
   (b) A module M satisfying the above conditions is called *Noetherian*. Give two different examples of Noetherian modules.

(4) Define the Heisenberg group $H_3(\mathbf{Z})$ as
   $$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbf{Z} \right\}$$
   with the operation of matrix multiplication.
   (a) Show that $H_3(\mathbf{Z})$ is nonabelian and compute the center of $H_3(\mathbf{Z})$.
   (b) Prove that $H_3(\mathbf{Z})$ is a nilpotent group.

(5) Let A be an abelian group, written additively. Call A *divisible* if for every $a \in A$ and nonzero $m \in \mathbf{Z}$ there exists $b \in A$ so that $a = mb$.
   (a) Can a nontrivial finite abelian group be divisible?
   (b) Show that the quotient of a divisible group is divisible.
   (c) Prove that $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ have no proper subgroups of finite index.

(6) (a) Prove that if $\mathbf{F}_q$ is a finite field with q elements, then there is a prime number p and a positive integer n such that $q = p^n$.
   (b) Prove that if $\mathbf{F}_q$ is a finite field with q elements, then its multiplicative group $\mathbf{F}_q^\times$ is cyclic.

(7) List all non-isomorphic finite abelian groups of order 24, and prove that your list is complete.

(8) (a) Compute the minimal polynomial $f(x)$ of $\alpha = \sqrt{2} + \sqrt[3]{5}$ over $\mathbf{Q}$.
  (b) Let E be the splitting field of the polynomial $f(x) = t^3 - 7 \in \mathbf{Q}[x]$. Compute the Galois group $\mathrm{Gal}(E/\mathbf{Q})$.

(9) Let E be the splitting field of the polynomial $f(t) = t^5 - 4t + 2$ over $\mathbf{Q}$. Compute the Galois group $\mathrm{Gal}(E/\mathbf{Q})$.

(10) Let R be a commutative ring with 1, and let A and B be left R-modules.
  (a) Sketch the construction of the tensor product $A \otimes_R B$.
  (b) Prove that $A \otimes_R B \cong B \otimes_R A$.

(11) Let G be a group, written multiplicatively, with identity e, and let S be a subset of G that generates G. For every element $x \in G \setminus \{e\}$, we denote by $\ell_S(x)$ the smallest integer k such that x can be written as a product of k elements of $S \cup S^{-1}$. We define $\ell_S(e) = 0$. The function $\ell_S : G \to \mathbf{N}_0$ is called the *word length function of G with respect to S*. The *growth function* of G with respect to a generating set S is the function $\gamma_S(n)$ that counts the number of elements of the group G of length at most n.
  (a) Prove that if S and T are finite sets that generate the group G, then there is a positive number c such that

$$\gamma_S(n) < c\gamma_T(n)$$

  for all sufficiently large n.
  (b) Compute the growth function for the free abelian group $\mathbf{Z}^2 = \mathbf{Z} \oplus \mathbf{Z}$ generated by the set $S = \{(1,0),(0,1)\}$.
  (c) Compute the growth function for the free group F_2 of rank 2 generated by the set $S = \{a, b\}$.

(12) (a) Define "category."
  (b) Prove that initial and terminal objects in a category are unique up to isomorphism.

(13) (a) Construct the regular representation of the cyclic group of order 4.
  (b) Construct a nontrivial irreducible representation of S_3.

(14) Prove that the subgroup of $F_2 = \langle a, b \rangle$ generated by the set $\{a^n b a^{-n} : n = 1, 2, 3, \dots\}$ is a free group of countably infinite rank.