ALGEBRA QUALIFYING EXAM
SEPTEMBER, 2012

Instructions: Do eight problems. Do not do more than eight problems.

(1) Let \( k \) be a field.
   (a) Prove the Division Algorithm for the polynomial ring \( k[x] \): If \( a \) and \( b \) are polynomials in \( k[x] \) with \( b \) nonzero, then there exist unique polynomials \( q, r \in k[x] \) such that
   \[
   a = qb + r
   \]
   and \( r = 0 \) or \( \deg r < \deg b \).
   (b) Use part (a) to show that \( k[x] \) is a principal ideal domain (PID).
   (c) Explain, with proof, whether \( k[x, y] \) is a PID.

(2) (a) Prove that a group of order 160 is not simple.
    (b) Prove that a group of order \( pqr \) for distinct primes \( p, q, r \) is not simple.

(3) Let \( R \) be a ring and let \( M \) be a left \( R \)-module.
   (a) Prove that the following three conditions are equivalent:
      (i) \( M \) satisfies the ascending chain condition on submodules.
      (ii) Every nonempty set of submodules of \( M \) contains a maximal element under inclusion.
      (iii) Every submodule of \( M \) is finitely generated.
   (b) A module \( M \) satisfying the above conditions is called Noetherian. Give two different examples of Noetherian modules.

(4) Define the Heisenberg group \( H_3(\mathbb{Z}) \) as
    \[
    \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}
    \]
    with the operation of matrix multiplication.
   (a) Show that \( H_3(\mathbb{Z}) \) is nonabelian and compute the center of \( H_3(\mathbb{Z}) \).
   (b) Prove that \( H_3(\mathbb{Z}) \) is a nilpotent group.

(5) Let \( A \) be an abelian group, written additively. Call \( A \) divisible if for every \( a \in A \) and nonzero \( m \in \mathbb{Z} \) there exists \( \overline{b} \in A \) so that \( a = m\overline{b} \).
   (a) Can a nontrivial finite abelian group be divisible?
   (b) Show that the quotient of a divisible group is divisible.
   (c) Prove that \( \mathbb{Q} \) and \( \mathbb{Q}/\mathbb{Z} \) have no proper subgroups of finite index.

(6) (a) Prove that if \( \mathbb{F}_q \) is a finite field with \( q \) elements, then there is a prime number \( p \) and a positive integer \( n \) such that \( q = p^n \).
   (b) Prove that if \( \mathbb{F}_q \) is a finite field with \( q \) elements, then its multiplicative group \( \mathbb{F}_q^\times \) is cyclic.

(7) List all non-isomorphic finite abelian groups of order 24, and prove that your list is complete.
(8) (a) Compute the minimal polynomial $f(x)$ of $\alpha = \sqrt{2} + \sqrt[3]{5}$ over $\mathbb{Q}$.
    (b) Let $E$ be the splitting field of the polynomial $f(x) = x^3 - 7 \in \mathbb{Q}[x]$.
        Compute the Galois group $\text{Gal}(E/\mathbb{Q})$.

(9) Let $E$ be the splitting field of the polynomial $f(t) = t^5 - 4t + 2$ over $\mathbb{Q}$.
    Compute the Galois group $\text{Gal}(E/\mathbb{Q})$.

(10) Let $R$ be a commutative ring with 1, and let $A$ and $B$ be left $R$-modules.
     (a) Sketch the construction of the tensor product $A \otimes_R B$.
     (b) Prove that $A \otimes_R B \cong B \otimes_R A$.

(11) Let $G$ be a group, written multiplicatively, with identity $e$, and let $S$ be a
     subset of $G$ that generates $G$. For every element $x \in G \setminus \{e\}$, we denote by
     $\ell_S(x)$ the smallest integer $k$ such that $x$ can be written as a product of $k$
     elements of $S \cup S^{-1}$. We define $\ell_S(e) = 0$. The function $\ell_S : G \to \mathbb{N}_0$
     is called the word length function of $G$ with respect to $S$. The growth function
     of $G$ with respect to a generating set $S$ is the function $\gamma_S(n)$ that counts
     the number of elements of the group $G$ of length at most $n$.
     (a) Prove that if $S$ and $T$ are finite sets that generate the group $G$, then
         there is a positive number $c$ such that:
         $$\gamma_S(n) < c \gamma_T(n)$$
         for all sufficiently large $n$.
     (b) Compute the growth function for the free abelian group $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$
         generated by the set $S = \{(1,0),(0,1)\}$.
     (c) Compute the growth function for the free group $F_2$ of rank 2 generated
         by the set $S = \{a,b\}$.

(12) (a) Define "category."
     (b) Prove that initial and terminal objects in a category are unique up to
         isomorphism.

(13) (a) Construct the regular representation of the cyclic group of order 4.
     (b) Construct a nontrivial irreducible representation of $S_3$.

(14) Prove that the subgroup of $F_2 = \langle a,b \rangle$ generated by the set
     $\{a^n b a^{-n} : n = 1, 2, 3, \ldots \}$ is a free group of countably infinite
     rank.