ALGEBRA QUALIFYING EXAM
MAY, 2011

PART 1

Instructions: Do four problems from Part 1. Do not do more than four problems from Part 1.

(1) Let p be an odd prime number, n a positive integer, and $q = p^n$. Let $\mathbf{F}_q$ be the finite field with q elements, and let $\mathbf{F}_q^\times = \mathbf{F}_q \setminus \{0\}$ be the multiplicative group of $\mathbf{F}_q$.
   (a) Prove that the group $\mathbf{F}_q^\times$ is cyclic.
   (b) Prove that
$$\prod_{a \in \mathbf{F}_q^\times} a = -1.$$

(2) (a) Find all cubic polynomials over the finite field $\mathbf{F}_2$ that are irreducible.
   (b) Construct the finite field $\mathbf{F}_8$ by computing the addition and multiplication tables for $\mathbf{F}_8$.

(3) (a) Prove that $E = \mathbf{Q}(\sqrt{3}, \sqrt{5})$ is a finite Galois extension of $\mathbf{Q}$, and compute the Galois group $\mathrm{Gal}(E/\mathbf{Q})$.
   (b) Find all subgroups of the $\mathrm{Gal}(E/\mathbf{Q})$, and compute the corresponding fixed fields.

(4) (a) Prove that the alternating group A_n is a normal subgroup of the symmetric group S_n.
   (b) Let N be a normal subgroup of S_n. Prove that if N contains a 3-cycle, then $N = A_n$ or $N = S_n$.

(5) Let $\mathbf{N}_0$ denote the nonnegative integers. A *Euclidean function* on an integral domain R is a function $N : R \setminus \{0\} \to \mathbf{N}_0$ such that, for all $a, b \in R$ with $b \neq 0$, then there exist $q, r \in R$ such that $a = qb + r$ and either $r = 0$ or $N(r) < N(b)$. An integral domain with a Euclidean function is called an *Euclidean domain.*
   (a) Prove that every Euclidean domain is a principal ideal domain.
   (b) Prove that the ring of Gaussian integers $\mathbf{Z}[i] = \{x + yi : x, y \in \mathbf{Z}\}$ is a Euclidean domain.

(6) (a) Show that the polynomial $x^n - p$ is irreducible in $\mathbf{Q}[x]$ for all positive integers n and all prime numbers p.
   (b) Let $f(x)$ and $g(x)$ be relatively prime irreducible polynomials in $\mathbf{Q}[x]$. Show that no root of $f(x)$ is a root of $g(x)$.

(7) A sequence of groups and group homomorphisms
$$N \xrightarrow{\alpha} G \xrightarrow{\beta} Q$$
is *exact at* G if the image of α is equal to the kernel of β. Let 1 denote the group whose only element is an identity. A *short exact sequence* is a

sequence of groups and group homomorphisms

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \longrightarrow 1$$

that is exact at N, G, and Q.

(a) Construct homomorphisms α and β such that

$$1 \longrightarrow \mathbf{Z}/3\mathbf{Z} \xrightarrow{\alpha} \mathbf{Z}/6\mathbf{Z} \xrightarrow{\beta} \mathbf{Z}/2\mathbf{Z} \longrightarrow 1$$

is a short exact sequence.

(b) Let S_3 be the symmetric group, that is, the group of permutations of $\{1, 2, 3\}$. Construct homomorphisms α and β such that

$$1 \longrightarrow \mathbf{Z}/3\mathbf{Z} \xrightarrow{\alpha} S_3 \xrightarrow{\beta} \mathbf{Z}/2\mathbf{Z} \longrightarrow 1$$

is a short exact sequence.

PART 2

Instructions: Do four problems from Part 2. Do not do more than four problems from Part 2.

(1) Find all groups of order 21 (up to isomorphism). Describe the isomorphism types as semidirect products $A \rtimes B$ of explicit groups A and B with explicit actions of B on A.

(2) Give the definition of a free abelian group. Show that $\mathbf{Q}$ is not a free group.

(3) Let $A = \mathbf{Z} \oplus \mathbf{Z}$ be the free abelian group on two generators and let B be the subgroup of A generated by $(3, 4)$.
(a) Find the rank of A/B.
(b) Express A/B as a direct sum of cyclic groups.
(c) Let C be the subgroup of A generated by $(9, 12)$. Express A/C as a direct sum of cyclic groups.

(4) Let R be a commutative ring with identity. Give the definition of a Noetherian module over the ring R. Give an example of a finitely generated R-module which is not Noetherian.

(5) Let $R = F[x, y]$, where F is a field. Prove that $Rx + Ry$ is NOT a cyclic submodule of R (viewed as a left R-module).

(6) Let S be a multiplicatively closed subset of a commutative ring R with identity and let M be a finitely generated R-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.