CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
FALL 2014
3 hours

Instructions. The exam consists of two parts. Choose a *total of seven problems*, including *at least three from each part.* Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Let R be a commutative ring with 1.
   a. Define the notion of a flat R-module.
   b. Show that the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is not flat. Is it projective?

2. Let G be a group of order 77.
   a. Prove that G is cyclic.
   b. Suppose that G acts on a set X with 17 elements. Prove that G fixes at least one point of X.

3. Let $\mathbb{F}_p$ be the field with p elements. Find the order of the group $G = \mathrm{SL}_2(\mathbb{F}_p)$ consisting of 2×2 matrices with entries in $\mathbb{F}_p$ and determinant one. Find the number of Sylow p-subgroups of G.

4. Prove that $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$.

5. Let R be a commutative ring with 1 and let I, J be ideals in R. Show that the canonical map $\phi : I \otimes_R J \to IJ$ given by sending $i \otimes j$ to ij is a surjective R-module homomorphism which need not be injective.

6. Show that the ring $\mathbb{Z}[\sqrt{-10}]$ is not a PID. Determine its group of units.

Part II

7. Let $\mathrm{GL}_n(\mathbb{Q})$ be the group of invertible $n \times n$ matrices with rational entries. An element g of $\mathrm{GL}_n(\mathbb{Q})$ is *torsion* if it has finite order.
   a. Prove that if g is a torsion element in $\mathrm{GL}_n(\mathbb{Q})$, then g is diagonalizable over $\mathbb{C}$.
   b. Suppose that g has prime order p in $\mathrm{GL}_n(\mathbb{Q})$. Prove that $p \leq n + 1$.

8. Let E/F be a finite extension of an infinite field F. We say that E/F is *simple* if $E = F(\alpha)$ for some α in E.
   a. Prove that E/F is simple if and only if the number of intermediate fields is finite.
   b. Give an example of a finite extension E/F that has infinitely many intermediate fields.

(Part II is continued on the next page.)

9. Let $f \in \mathbb{Q}[x]$ be a monic irreducible polynomial of degree 3 and let K be the splitting field of f over $\mathbb{Q}$.
   a. Determine all possible Galois groups $\mathrm{Gal}(K/\mathbb{Q})$ and prove that these are all such.
   b. Give an explicit example for each possible Galois group.

10. Denote the field with p elements by $\mathbb{F}_p$. For each of the fields F below, let $f(x) = x^6 - 3$ in $F[x]$ and let K be the splitting field of f over F. Find the degree $[K:F]$.
    a) $F = \mathbb{Q}(\sqrt{-3})$ b) $F = \mathbb{F}_7$ c) $F = \mathbb{F}_5$.

11. Let K be the splitting field over $\mathbb{Q}$ of $x^8 - 1$.
    a. Find the degree $[K:\mathbb{Q}]$ and describe the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ as an abstract group.
    b. Find explicitly all subgroups of G and the corresponding subfields of K under the Galois correspondence.

12. Let k be an algebraically closed field and let $\mathbb{A}^n = k^n$ denote the affine space of dimension n. If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Part II

7. Let V be a finite dimensional vector space over $\mathbb{Q}$ and let $T\colon V \to V$ be a linear transformation such that $T^2 = -1$. Suppose V has a **non-trivial proper** subspace W invariant under T. What is the smallest possible value for $\dim_{\mathbb{Q}} V$?

8. Let K be a finite extension field of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in K.

9. Let $\mathbb{F}_p$ denote the field with p elements. Determine the Galois group of the splitting field of $x^3 - x + 1$ over the following fields: a) $\mathbb{F}_3$ b) $\mathbb{F}_5$ c) $\mathbb{Q}$.
(Recall that the discriminant $D(f)$ of $f(x) = x^3 + bx + c$ is given by $D(f) = -4b^3 - 27c^2$.)

10. Let K be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(x) = x^6 + 3$.
    a. Show that K contains a primitive sixth root of unity.
    b. Show that K is Galois over $\mathbb{Q}$.
    c. Determine the number of fields F of degree 3 over $\mathbb{Q}$ and contained in K.

11. Let K/F be a finite extension of finite fields. Prove that the norm map $N_{K/F}\colon K \to F$ is surjective.

12. Let k be an algebraically closed field and $\mathbb{A}^n = k^n$ denote the affine space of dimension n. Consider the zero set
$$Y = \mathcal{Z}(x^2 - yz,\, xz - x) \subset \mathbb{A}^3.$$
Decompose Y into irreducible components.