CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2013
3 hours

Instructions. The exam consists of two parts. Choose a *total of eight problems*, including *at least three from each part.* Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. The class equation for a finite group G relates its order to the indices of centralizers of elements of G. Let p be a prime.
   a. Suppose $|G| = p^m$ with $m \geq 1$. Use the class equation to show that the center $Z(G)$ is nontrivial.
   b. If $|G| = p^2$, show that G is abelian.
   c. For all primes p, construct a non-abelian group of order p^3 as a semidirect product $N \rtimes H$. Specify the action of H on N.

2a. For $n \geq 3$, identify the group of symmetries in $\mathbb{R}^3$ of the regular n-gon. Justify that you found all the symmetries.

b. The *rigid motions* of a 3-dimensional solid are its rotational symmetries in $\mathbb{R}^3$. These orientation preserving isometries form a group under composition. Show that the group of rigid motions of the tetrahedron is isomorphic to the alternating group A_4.

3a. Denote the number of Sylow p-subgroups of a finite group G by $n_p(G)$.

i. Determine $(n_2(G), n_3(G))$ when G is the cyclic group $\mathbb{Z}/12\mathbb{Z}$, the alternating group A_4 and the dihedral group D with 12 elements.

ii. Show that no other pair $(n_2(G), n_3(G))$ occurs if G is a group of order 12.

b. Prove that no group of order 351 is simple by considering Sylow p-subgroups.

4. Let R be a commutative ring with identity and let I_1, I_2, $\ldots I_k$ be ideals in R.
   a. Assume that the map $R \to R/I_1 \times R/I_2 \times \cdots \times R/I_k$ given by $r \mapsto (r + I_1, r + I_2, \ldots, r + I_k)$ is a ring homomorphism and find its kernel.
   b. Give the condition from the Chinese Remainder Theorem for this map to be surjective.
   c. Let $n = p_1 \cdots p_k$ be the product of distinct odd primes $p_1, \ldots, p_k$. How many distinct solutions are there modulo n to the congruence $x^2 \equiv 1 \bmod n$?

5. Let R be a commutative ring with unity.
   a. Define principal, prime and maximal ideals of R.
   b. Sketch a proof that every maximal ideal of R is prime.
   c. Give a detailed proof that every non-zero prime ideal in a principal ideal domain is maximal.
   d. Give an example of a commutative ring R and a non-zero prime ideal of R which is not maximal.

6. Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity in $\mathbb{C}$. The ring of Eisenstein Integers is
$$\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}.$$
   a. Prove that $\mathbb{Z}[\omega]$ is a Euclidean Domain with respect to the norm $N(a+b\omega) = a^2 - ab + b^2 = |a+b\omega|^2$.
   b. Find all the units in $\mathbb{Z}[\omega]$ and prove that your list is complete.

Part II

7. Consider the polynomial $f(x) = x^5 - 25x - 15$.
 a. Show that f is irreducible over $\mathbb{Q}$.
 b. Factor f as much as possible over the finite fields $\mathbb{F}_2$ and $\mathbb{F}_3$.
 c. Determine the Galois group $\mathrm{Gal}_{\mathbb{Q}}(f)$ of a splitting field of f over $\mathbb{Q}$. State clearly any important facts you are using from group theory.

8a. Let $\boldsymbol{\mu}_n$ be the group of n^{th} roots of unity, i.e. the roots of $x^n - 1$ in a splitting field over a field F of characteristic 0. Prove that the Galois group $\mathrm{Gal}(F(\boldsymbol{\mu}_n)/F)$ is abelian.
 b. Determine the Galois group $\mathrm{Gal}_{\mathbb{Q}}(g)$ of $g(x) = x^3 - 5$ over $\mathbb{Q}$.
 c. Prove that $\sqrt[3]{5}$ is not contained in any cyclotomic field $\mathbb{Q}(\boldsymbol{\mu}_n)$.

9. Let K be the splitting field of $f(x) = x^n - 1$ over the field $F = \mathbb{F}_p$ with p elements, where p is a prime not dividing n.
 a. Let d be the smallest positive integer such that $p^d \equiv 1 \pmod{n}$. Prove that $[K:F] = d$.
 b. Determine the Galois group $\mathrm{Gal}_F(f)$ of $f(x) = x^{31} - 1$ over $F = \mathbb{F}_{101}$.

10. Let $R \neq 0$ be a commutative ring with unity. Recall that the radical of an ideal I of R is defined as:

$$\mathrm{rad}\, I = \{r \in R \,|\, r^n \in I \text{ for some positive integer } n, \text{ which may depend on } r\}.$$

 a. Prove that $\mathrm{rad}\,(0)$ is an ideal of R.
 b. Prove that if a is in $\mathrm{rad}\,(0)$, then $1 - a$ is invertible.
  Suggestion. Consider a geometric series for the inverse.
 c. Let the ring R be Noetherian. Prove that $\mathrm{rad}\,(0) = \bigcap P$, where the intersection is taken over all prime ideals P of R.
  Suggestion. Given an element $a \in R$, not in $\mathrm{rad}\,(0)$, consider the set of those ideals J of R such that no positive power of a is in J. Use this to construct a prime ideal not containing a.

11a. Prove that if $d = \gcd(m, n)$, then $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \simeq \mathbb{Z}/d$.
 b. Give an example of a short exact sequence of non-zero R-modules $0 \to M_1 \to M \to M_2 \to 0$ over some commutative ring R and another R-module N, such that the sequence obtained by applying the functor $\mathrm{Hom}(\text{-}\,\text{-}, N)$ is no longer exact.

12a. List the different abelian groups of order 72 up to isomorphism.
 b. Let $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and let $T\colon\ M \to M$ be given by

$$T(x, y, z) = (4x + 2z, 2y, 2x + 10z).$$

 Show that the cokernel $M/T(M)$ is an abelian group of order 72 and determine its isomorphism class in your list from part a.