CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM
SPRING 2014
3 hours

Instructions. The exam consists of two parts. Choose a *total of eight problems*, including *at least three from each part.* Indicate on the front cover of your answer book the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part I

1. Let $f(x) = x^3 - 5x^2 + 10x + 20$ in $\mathbb{Z}[x]$ and let $(f) = f\mathbb{Z}[x]$ be the principal ideal generated by f. Let $\mathbb{Z}[\sqrt{3}]$ be the quadratic integer ring $\{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$. Let R be the ring $\mathbb{Z}[\sqrt{3}] \otimes_{\mathbb{Z}} (\mathbb{Z}[x]/f\mathbb{Z}[x])$. Prove or disprove that R is an integral domain.

2. Let p and q be distinct primes, $R = \mathbb{Z}/pq\mathbb{Z}$, x an indeterminate over R and S the ring $R[x]/x^3R[x]$. Determine:
   a. all the maximal ideals of S;
   b. all the prime ideals of S;
   c. the nilradical of S.

3. Let G be the group $\mathbb{Z}/35\mathbb{Z}$ and H the group $\mathbb{Z}/15\mathbb{Z}$.
   a. Determine all isomorphism classes of semidirect product groups $G \rtimes_{\varphi} H$, where φ is a homomorphism from H to $\operatorname{Aut} G$.
   b. With G and H as above, give an example of a group K such that the following sequence is exact, but does not split: $0 \to G \to K \to H \to 0$.

4. Prove that a group of order 132 is not simple.

5. For each of the following, either construct the object or prove it does not exist:
   a. an integral domain consisting of 15 elements;
   b. an integral domain with non-zero characteristic and infinitely many elements.

6. Let R be the quadratic integer ring $R := \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$. Prove that R is a Euclidean domain with respect to the norm $N(a + b\sqrt{-2}) = (a + b\sqrt{-2})(a - b\sqrt{-2})$.

(see next page for Part II)

Part II

7. Let V be a finite dimensional vector space over $\mathbb{Q}$ and let $T\colon V \to V$ be a linear transformation such that $T^2 = -1$. Suppose V has a **non-trivial proper** subspace W invariant under T. What is the smallest possible value for $\dim_{\mathbb{Q}} V$?

8. Let K be a finite extension field of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in K.

9. Let $\mathbb{F}_p$ denote the field with p elements. Determine the Galois group of the splitting field of $x^3 - x + 1$ over the following fields: a) $\mathbb{F}_3$ b) $\mathbb{F}_5$ c) $\mathbb{Q}$.

   (Recall that the discriminant $D(f)$ of $f(x) = x^3 + bx + c$ is given by $D(f) = -4b^3 - 27c^2$.)

10. Let K be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(x) = x^6 + 3$.
    a. Show that K contains a primitive sixth root of unity.
    b. Show that K is Galois over $\mathbb{Q}$.
    c. Determine the number of fields F of degree 3 over $\mathbb{Q}$ and contained in K.

11. Let K/F be a finite extension of finite fields. Prove that the norm map $N_{K/F}\colon K \to F$ is surjective.

12. Let k be an algebraically closed field and $\mathbb{A}^n = k^n$ denote the affine space of dimension n. Consider the zero set

$$Y = \mathcal{Z}(x^2 - yz,\, xz - x) \subset \mathbb{A}^3.$$

    Decompose Y into irreducible components.