QUALIFYING EXAM FOR COMPLEX ANALYSIS

FALL 2005
DO ANY 8 PROBLEMS.
SHOW ALL WORK TO RECEIVE FULL CREDIT.

Problem 0.1. (1) Let f be a bounded holomorphic function from $\mathbb{C}$ to $\mathbb{C}$. Show f is constant.

(2) Let g_n be a sequence of holomorphic functions on $\mathbb{D}$ converging uniformly to a function f. Show that the derivatives g_n' also converge. Deduce that f is holomorphic.

Problem 0.2. Calculate the integral:

$$\int_0^\infty \frac{\sin(x)}{x(1+x^2)}dx.$$

Carefully justify all your steps.

Problem 0.3. Let $u(x,y) = (x+1)y$.

(1) Show u is harmonic on the entire plane.
(2) Find a conjugate harmonic function $v(x,y)$ to u.
(3) Write down a holomorphic function f with $Re(f) = u$ and $Im(f) = v$.

Problem 0.4. Show that if f is a holomorphic function with $f'(z_0) = 0$ that f is not one-to-one in a neighborhood of z_0.

Problem 0.5. If $f(z)$ is an entire function without any roots prove that there is an entire function $g(z)$ with $f(z) = e^{g(z)}$.

Problem 0.6. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. (Hints: Use Rouché's theorem.)

Problem 0.7. Let u be harmonic on a connected domain Ω in $\mathbb{C}$.

(1) Show that $f = u_x - iu_y$ is holomorphic in Ω.
(2) Suppose u is the real part of a holomorphic function g, show $g' = f$.
(3) Give necessary and sufficient conditions on Ω such that every harmonic function on Ω is the real part of a holomorphic function. Justify your answer.

Problem 0.8. (1) State Schwartz's Lemma.

(2) Find all holomorphic automorphisms of the unit disk. Justify your answer using Schwartz's Lemma.

Problem 0.9. Suppose Ω is a simply connected domain in $\mathbb{C}$ and g is a holomorphic function on Ω which is not identically zero. Given a positive integer n, show that g has a holomorphic nth root on Ω if and only if every zero of g has multiplicity divisible by n.

Problem 0.10. Let $\mathbb{D}$ be the unit disk and E any compact subset of $\mathbb{D}$. Use the Poisson formula to prove the following form of Harnack's inequality: there is a constant M depending only on E such that for any positive harmonic function u on Ω, and any $z_1, z_2 \in E$ we have $u(z_1) \leq M u(z_2)$.

Do not quote Harnack's inequality in your proof.

Problem 0.11. Suppose $f(z)$ is analytic on $|z| < 1$ and continuous on $|z| \leq 1$. Assume $f(z) = 0$ on an arc of the circle $|z| = 1$. Prove that $f(z) \equiv 0$.

Problem 0.12. Suppose that the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence R with $0 < R < \infty$ and that f has an extension which has a pole on the boundary of the disk of radius R. Show that the series above diverges at all points of the boundary of the disk.

Problem 0.13. Let Ω_1 be a region in the complex plane. A part LMN of the boundary of Ω_1 is on the real axis. Suppose that $F_1(z)$ is analytic in the region Ω_1 and that $F_1(z)$ assumes real values on the part LMN of the real axis. Prove the Schwarz's reflection principle which says that the analytic continuation of F_1 into the reflection region Ω_2 of Ω_1 with LMN as the mirror is given by

$$F_2(z) = \overline{F_1(\overline{z})}$$

Problem 0.14. Suppose that f is a holomorphic map from the upper half plane onto itself such that f is invertible. Show that f has to be of the form $(az+b)/(cz+d)$ where a,b,c and d are real.