QUALIFYING EXAM FOR COMPLEX ANALYSIS

FALL 2005
DO ANY 8 PROBLEMS.
SHOW ALL WORK TO RECEIVE FULL CREDIT.

Problem 0.1. (1) Let $f$ be a bounded holomorphic function from $\mathbb{C}$ to $\mathbb{C}$. Show $f$ is constant.
(2) Let $g_n$ be a sequence of holomorphic functions on $\mathbb{D}$ converging uniformly to a function $f$. Show that the derivatives $g'_n$ also converge. Deduce that $f$ is holomorphic.

Problem 0.2. Calculate the integral:

$$\int_{0}^{\infty} \frac{\sin(x)}{x(1+x^2)} dx.$$ 

Carefully justify all your steps.

Problem 0.3. Let $u(x, y) = (x + 1)y$.
(1) Show $u$ is harmonic on the entire plane.
(2) Find a conjugate harmonic function $v(x, y)$ to $u$.
(3) Write down a holomorphic function $f$ with $Re(f) = u$ and $Im(f) = v$.

Problem 0.4. Show that if $f$ is a holomorphic function with $f''(z_0) = 0$ that $f$ is not one-to-one in a neighborhood of $z_0$.

Problem 0.5. If $f(z)$ is an entire function without any roots prove that there is an entire function $g(z)$ with $f(z) = e^{g(z)}$.

Problem 0.6. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. (Hints: Use Rouché’s theorem.)

Problem 0.7. Let $u$ be harmonic on a connected domain $\Omega$ in $\mathbb{C}$.
(1) Show that $f = u_x - iu_y$ is holomorphic in $\Omega$.
(2) Suppose $u$ is the real part of a holomorphic function $g$, show $g' = f$.
(3) Give necessary and sufficient conditions on $\Omega$ such that every harmonic function on $\Omega$ is the real part of a holomorphic function. Justify your answer.

Problem 0.8. (1) State Schwartz’s Lemma.
(2) Find all holomorphic automorphisms of the unit disk. Justify your answer using Schwartz’s Lemma.

**Problem 0.9.** Suppose $\Omega$ is a simply connected domain in $\mathbb{C}$ and $g$ is a holomorphic function on $\Omega$ which is not identically zero. Given a positive integer $n$, show that $g$ has a holomorphic $n$th root on $\Omega$ if and only if every zero of $g$ has multiplicity divisible by $n$.

**Problem 0.10.** Let $\mathbb{D}$ be the unit disk and $E$ any compact subset of $\mathbb{D}$. Use the Poisson formula to prove the following form of Harnack’s inequality: there is a constant $M$ depending only on $E$ such that for any positive harmonic function $u$ on $\Omega$, and any $z_1, z_2 \in E$ we have $u(z_1) \leq M u(z_2)$.

*Do not quote Harnack’s inequality in your proof.*

**Problem 0.11.** Suppose $f(z)$ is analytic on $|z| < 1$ and continuous on $|z| \leq 1$. Assume $f(z) = 0$ on an arc of the circle $|z| = 1$. Prove that $f(z) \equiv 0$.

**Problem 0.12.** Suppose that the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R$ with $0 < R < \infty$ and that $f$ has an extension which has a pole on the boundary of the disk of radius $R$. Show that the series above diverges at all points of the boundary of the disk.

**Problem 0.13.** Let $\Omega_1$ be a region in the complex plane. A part $LMN$ of the boundary of $\Omega_1$ is on the real axis. Suppose that $F_1(z)$ is analytic in the region $\Omega_1$ and that $F_1(z)$ assumes real values on the part $LMN$ of the real axis. Prove the Schwarz’s reflection principle which says that the analytic continuation of $F_1$ into the reflection region $\Omega_2$ of $\Omega_1$ with $LMN$ as the mirror is given by

$$F_2(z) = \overline{F_1(\bar{z})}$$

**Problem 0.14.** Suppose that $f$ is a holomorphic map from the upper half plane onto itself such that $f$ is invertible. Show that $f$ has to be of the form $(az + b)/(cz + d)$ where $a, b, c$ and $d$ are real.