Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
September, 2014

Notations

- $\mathrm{Re}(z)$: the real part of a complex number z
- $\mathrm{Im}(z)$: the imaginary part of a complex number z
- $\mathbb{C}$: the complex plane
- $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: the Riemann sphere
- $\Delta = \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $\mathrm{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: the upper half-plane
- $\mathcal{O}(\Omega)$: the set of all holomorphic functions defined on a domain Ω

PART I: Do Any TWO Problems.

(1) (a) Find the number of zeros of $f(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$.
(b) Derive the Fundamental Theorem of Algebra as a corollary of Rouché's theorem.

(2) (a) State the Riemann mapping theorem.
(b) Prove the uniqueness part of the theorem.
(c) Find the Riemann map for the wedge domain $\{z = x + iy \,|\, 0 < x < y\}$.

(3) If f is a meromorphic function on an open set G, show that there exist holomorphic functions g and h on G such that $f = g/h$.

(4) Let Ω be an open subset in the complex plane and f be a holomorphic function defined on Ω. Suppose that $f'(z_0) = 0$ at a point $z_0 \in \Omega$. Show that f is not injective on any neighborhood of z_0.

PART II: Do Any TWO Problems.

(1) Let a, b, c, d be four points on the unit circle S^1 arranged in counterclockwise order. Consider the cross-ratio $cr(\{a,b,c,d\}) = \frac{(b-a)(d-c)}{(c-b)(d-a)}$.
(a) Show that $cr(\{a,b,c,d\})$ is always a positive real number.
(b) Show that $cr(\{a,b,c,d\})$ is a constant if and only if the hyperbolic distance between the geodesic connecting a to b and the one connecting c to d is a constant.

(2) Show that each element f of $\mathrm{Aut}(\Delta)$ can be expressed as

$$f(z) = e^{i\theta}\frac{z - \alpha}{1 - \overline{\alpha}z},$$

for a constant $0 \leq \theta < 2\pi$, where $\alpha = f^{-1}(0)$ and $z \in \Delta$.

(3) (a) Let f be holomorphic on $G = \{z : \mathrm{Re}(z) > 0\}$ with $\mathrm{Re}f(z) > 0$ for all $z \in G$, and $f(a) = a$ for some $a \in G$. Show that $|f'(a)| \leq 1$.

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(b) Suppose f is holomorphic in the annulus $1 \le |z| \le 2$ and that $|f| \le 1$ for $|z| = 1$, $|f| \le 4$ for $|z| = 2$. Prove that $|f(z)| \le |z|^2$ for all z in the annulus.

PART III: Do Any FOUR Problems.

(1) (a) Give the Cauchy Integral Formula for derivatives.

(b) Suppose that $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ is bounded by 1 on $|z| \le 1$. Show that

(i) $|a_k| \le 1$ for each $0 \le k \le n$,

(ii) $|P_n(z)| \le |z|^n$ on $|z| \ge 1$, and

(iii) $P_n(z) = a_k z^k$ with $|a_k| = 1$ if P_n maps the unit circle $\{z : |z| = 1\}$ into itself.

(2) Evaluate the definite integral

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(1+x^2)^2}$$

(3) Let G be a simply connected region and let $f \in \mathcal{O}(G)$ such that $f(z) \neq 0$ for any z in G. Show that there exists a holomorphic function g in G such that $f(z) = e^{g(z)}$ for all z in G.

(4) Let G be a region. Show if every harmonic function defined on G can be expressed as the real part of a holomorphic function f on G, then G is simply connected.

(5) Show that

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2}\right]$$

for all $z \in \mathbb{C}$ and that the convergence is uniform on any compact subset of $\mathbb{C}$.

(6) (a) State the Monodromy Theorem.

(b) Let G be a region and f and g be two holomorphic functions defined on G. Suppose that there exists a point $z_0 \in G$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all nonnegative integer n. Show that $f = g$.

(7) Given a lattice

$$L = \{m\omega_1 + n\omega_2,\ \omega_1, \omega_2 \in \mathbb{C}, \operatorname{Im}(\frac{\omega_2}{\omega_1}) \neq 0,\ m, n \in \mathbb{Z}\},$$

prove that there exist $\eta_1, \eta_2 \in L$ such that every element in L can be written as $m'\eta_1 + n'\eta_2$, $m', n' \in \mathbb{Z}$ and $\tau = \eta_2/\eta_1$ satisfies

$$-\frac{1}{2} < \operatorname{Re}(\tau) \le \frac{1}{2}, \quad |\tau| \ge 1, \text{ and if } |\tau| = 1, \operatorname{Re}(\tau) \ge 0.$$

(8) (a) Let f be holomorphic and injective in $\{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that 0 cannot be an essential singularity of f.

(b) Show that $\operatorname{Aut}(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$.