Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
Spring 2014

Notation:

- $\operatorname{Im}(z)$: the imaginary part of a complex number z
- $\mathbb{C}$: the complex plane
- $\Delta = \{z \in \mathbb{C} : |z| < 1\}$: the open unit disk
- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$: the unit circle
- $\operatorname{Aut}(\Omega)$: the group of all conformal automorphisms of a domain Ω
- $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$: the upper half-plane
- $\mathcal{O}(\Omega)$: the set of all holomorphic functions defined on a domain Ω

PART I: Do Any TWO Problems.

(1) (a) Find the number of zeros of $f(z) = z^6 - 5z^4 + 3z^2 - 1$ in $|z| < 1$.
(b) Let G be a region. Show that if $\mathcal{F} \subset \mathcal{O}(G)$ is normal, then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also normal.

(2) Evaluate the definite integral

$$\int_0^\infty \frac{dx}{x^4 + x^2 + 1}$$

(3) State and prove the Maximum Modulus Theorem for holomorphic functions.

(4) Suppose Ω is a region, $f_n \in \mathcal{O}(\Omega)$ for $n = 1, 2, 3, \cdots$, each f_n is one-to-one in Ω, and $f_n \to f$ uniformly on any compact subset of Ω. Prove that f is either constant or one-to-one in Ω.

PART II: Do Any TWO Problems.

(1) Let a, b, c, d be four points in the complex plane $\mathbb{C}$. Consider the cross ratio $cr(\{a, b, c, d\}) = \frac{(b-a)(d-c)}{(b-c)(d-a)}$.
(a) Show that $cr(\{a, b, c, d\})$ real if and only if the four points lie on a circle or straight line.
(b) Show that if the four points lie on the unit circle S^1 such that the chord connecting a, b intersects the chord connecting c, d then $cr(\{a, b, c, d\})$ is positive; moreover it is equal to $+2$ if and only if the geodesic connecting a, b orthogonally intersects the geodesic connecting c, d.

(2) State and prove the Schwarz Lemma.

(3) Let $f : \mathbb{H} \to \mathbb{H}$ be holomorphic. Prove:
(a) $\frac{|f(z)-f(z_0)|}{|f(z)-\overline{f(z_0)}|} \leq \frac{|z-z_0|}{|z-\bar{z}_0|}$ for any two different points $z, z_0 \in \mathbb{H}$.
(b) $\frac{|f'(z)|}{\operatorname{Im}(f(z))} \leq \frac{1}{\operatorname{Im}(z)}$ for any $z \in \mathbb{H}$.

PART III: Do Any FOUR Problems.

(1) Find the Riemann map for the half unit disk $\{z = x+iy : |z| < 1 \text{ and } x < 0\}$.

(2) Let S denote the class of all $f \in \mathcal{O}(\Delta)$ that are one-to-one in Δ and satisfy $f(0) = 0$ and $f'(0) = 1$. Prove:

(a) If $f \in S$, $|\alpha| = 1$ and $g(z) = \overline{\alpha} f(\alpha z)$, then $g \in S$.

(b) If $f \in S$ there exists a $g \in S$ such that $g^2(z) = f(z^2)$ for all $z \in \Delta$.

(3) Suppose Ω is a region, L is a straight line or a circular arc, $\Omega - L$ is the union of two regions Ω_1 and Ω_2, f is continuous in Ω, and f is holomorphic in Ω_1 and Ω_2. Show that f is holomorphic in Ω.

(4) (a) Let f be holomorphic and injective in $\{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that 0 cannot be an essential singularity of f.

(b) Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Find $\mathrm{Aut}(\mathbb{C}^*)$ and prove your result.

(5) Show that for each $R > 0$, if n is large enough,

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \ldots \frac{z^n}{n!} \text{ has } \textit{no} \text{ zeros in } |z| < R.$$

(6) Show that the function

$$w(z) = \rho z + \frac{1}{\rho z}, \ \rho > 0$$

maps the unit disk Δ one-to-one and onto the exterior of an ellipse whose major and minor axes have the ratio

$$\frac{\rho + \frac{1}{\rho}}{|\rho - \frac{1}{\rho}|}.$$

(7) Let $\{u_n\}$ be a sequence of harmonic functions in a region V. Prove:

(a) If $u_n \to u$ uniformly on compact subsets of V, then u is harmonic in V.

(b) If $u_1 \leq u_2 \leq u_3 \leq \cdots$, then either $\{u_n\}$ converges uniformly on any compact subset of V or $u_n(z) \to \infty$ for every $z \in V$.

(8) Given the equation

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3$$

with roots e_1, e_2, e_3, assume that g_2, g_3 are real and the discriminant satisfies $g_2^3 - 27g_3^2 > 0$.

(a) Show that the e_i, $i = 1, 2, 3$, are real.

(b) Using (a), assume that $e_1 > e_2 > e_3$. Let

$$\omega_1 = \int_{e_1}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} dt$$

and

$$\omega_3 = i \int_{-\infty}^{e_3} (g_3 + g_2 t - 4t^3)^{-1/2} dt.$$

Show that ω_1 is real and $\omega_3 = \omega_2 + \omega_1$ is purely imaginary.