COMPLEX VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work all problems. Time: three hours.

September 5, 2000

1. Suppose u is harmonic in a neighborhood of $|z| \leq 1$. Prove Poisson's formula, that for w in the open unit disk,
$$u(w) = \frac{1}{2\pi} \int_{|z|=1} u(z) \frac{1 - |w|^2}{|z - w|^2} \, |dz| \, .$$

2. Show that a fractional linear transformation is determined by the image of any three distinct points.

3. Find a conformal map of the quarter disk
$$\{z \ : \ \text{Re}z > 0, \ \text{Im}z > 0, \ |z| < 1\}$$
onto the unit disk.

4. Prove that a harmonic function of a holomorphic function is harmonic.

5. Prove that any biholomorphic map of the unit disk onto itself is of the form $z \to e^{i\theta}(z - \alpha)/(1 - \overline{\alpha} z)$, with θ real and $|\alpha| < 1$.

6. Show that a family of holomorphic functions which is bounded on a domain D is equicontinuous on any compact subset of D.

7. Use the calculus of residues to evaluate $\int_{-\infty}^{\infty} \frac{\cos 2x}{1+2x^2+x^4} \, dx$.

8. Suppose $f_1(z), f_2(z), \ldots$ is a sequence of holomorphic functions defined on a domain D for which $\sum_1^\infty |f_k(z)|$ converges uniformly on compact subsets of D. Show that $\prod_1^\infty (1 - f_k(z))$ converges uniformly on compact subsets of D to a holomorphic function $g(z)$ in D, and that $g(z)$ vanishes exactly at those points for which at least one of the factors vanishes.