Complex Variables Qualifying Exam

Spring, 1999

Do Any Eight Problems.

Throughout, $D$ denotes the disc $\{z : |z| < 1\}$.

1. Find a conformal map from the strip $S = \{z : 0 < \text{Re}z < 1\}$ onto $D$.

2. Use the calculus of residues to compute

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(1 + x^2)^{3/2}} dx$$

3. Let $I$ denote the closed interval $[0, 1]$.

   a) Show there exists a unique single-valued analytic branch $\varphi(z)$ of
      $\log(z^{1/2} - 1)$ in $C - I$ with $\varphi(2)$ real.

   b) Find the Laurent expansion of $\varphi$ in the annulus $\{z : 1 < |z| < \infty\}$.

4. Let $B = \{z : |\text{Re}z| < 1/2$ and $|\text{Im}z| < 1/2\}$. Suppose $f$ is analytic on $B$
and maps $B$ into $D$. Show

$$|f'(0)| \leq \frac{8}{\pi}.$$  

For extra credit ($\frac{1}{2}$ problem), show

$$|f'(0)| \leq 2.$$  

5. Suppose $f$ and $g$ are entire analytic functions. Suppose that there exist

$z_0 \in \mathbb{C}, C > 0$ and $\alpha > 0$, $\alpha$ not an integer, such that

$$|f(z)| \leq C|z - z_0|^\alpha |g(z)|$$

for all $z \in \mathbb{C}$. Show that $f(z) \equiv 0$.  

6. For \( n = 0, 1, 2, 3, \ldots \) define \( P_n(z) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} z^{2k} \). Show there exists \( n_0 \) such that, for all \( n \geq n_0 \), \( P_n(z) \) has no zeros in \( D \).

For extra credit (1 problem), show \( n_0 = 0 \) will do.

7. Suppose \( f \) is analytic on \( D \), and, in fact, maps \( D \) into \( D \), with \( f(0) = 0 \), \( f'(0) = a \), and \( |a| < 1 \). Let \( M_a \) denote the Möbius transformation

\[
M_a(w) = \frac{w + a}{1 + \overline{a}w}.
\]

Show there exists a function \( h(z) \), analytic on \( D \), with \( |h(z)| \leq 1 \) for all \( z \in D \) and \( h(0) = 0 \), such that

\[
f(z) = z M_a(h(z)) \quad \text{for all} \quad z \in D.
\]

8. Suppose \( f \) is analytic and bounded in \( D \) and let \( M = \sup_{z \in D} |f(z)| \).

Suppose further that \( f(0) = 1 \). Let \( n(r) \) denote the number of zeros of \( f \) in the disc \( \{ z : |z| < r \} \), a zero of multiplicity \( m \) being counted \( m \) times. Show

\[
n(r) \leq \frac{\ln M}{\ln(1/r)}.
\]

9. Suppose \( u \) is a non-constant harmonic function on the region \( G \) in \( \mathbb{C} \). Let

\[
G_0 = \{ z \in G : \frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0 \}.
\]

Show \( G_0 \) has no points of accumulation in \( G \), i.e., for every \( z_0 \in G \), there exists \( \epsilon > 0 \), such that \( 0 < |z - z_0| < \epsilon \) implies \( z \) is not in \( G \).

10. Let \( f \) be analytic in a region \( G \) and for \( w \in \mathbb{C} \), let \( \mu(w) \) be the number of zeros in \( G \) of \( f(z) - w \), counted according to multiplicity. Suppose \( \mu(w_0) \geq n \), where \( w_0 \in \mathbb{C} \) and \( n \) is a positive integer. Show there exists a neighborhood \( W \) of \( w_0 \) such that every \( w \in W - \{ w_0 \} \) has at least \( n \) distinct pre-images under \( f \) in \( G \).

11. Given \( f_n \) analytic, mapping \( D \) in \( D \), \( n = 1, 2, 3, \ldots \). Let \( A = \{ \frac{1}{k} : k = 2, 3, 4, \ldots \} \cup \{ 0 \} \). Suppose for every \( a \in A \), that \( \lim_{n \to \infty} f_n(a) \) exists in \( \mathbb{C} \).

Show there exists a complex-valued function \( f \) on \( D \) such that \( \lim_{n \to \infty} f_n(z) = f(z) \) for all \( z \in D \), the convergence being uniform on every compact subset of \( D \).