Complex Variables Qualifying Exam

Spring, 1999

Do Any Eight Problems.

Throughout, $\mathbb{D}$ denotes the disc $\{z : |z| < 1\}$.

1. Find a conformal map from the strip $S = \{z : 0 < \text{Re}z < 1\}$ onto $\mathbb{D}$.

2. Use the calculus of residues to compute

$$\int_{-\infty}^{\infty} \frac{e^{ix}dx}{(1+x^2)^2}.$$

3. Let I denote the closed interval $[0, 1]$.

a) Show there exists a unique single-valued analytic branch $\varphi(z)$ of $\log(\frac{z-1}{z})$ in $\mathbb{C} - I$ with $\varphi(2)$ real.

b) Find the Laurent expansion of φ in the annulus $\{z : 1 < |z| < \infty\}$.

4. Let $B = \{z : |\text{Re}z| < 1/2 \text{ and } |\text{Im}z| < 1/2\}$. Suppose f is analytic on B and maps B into $\mathbb{D}$. Show

$$|f'(0)| \leq \frac{8}{\pi}.$$

For extra credit ($\frac{1}{2}$ problem), show

$$|f'(0)| \leq 2.$$

5. Suppose f and g are entire analytic functions. Suppose that there exist $z_0 \in \mathbb{C}, C > 0$ and $\alpha > 0$, α not an integer, such that

$$|f(z)| \leq C|z - z_0|^{\alpha}|g(z)|$$

for all $z \in \mathbb{C}$. Show that $f(z) \equiv 0$.

6. For $n = 0, 1, 2, 3, ...$ define $P_n(z) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} z^{2k}$. Show there exists n_0 such that, for all $n \geq n_0$, $P_n(z)$ has no zeros in $\mathbb{D}$.

For extra credit (1 problem), show $n_0 = 0$ will do.

7. Suppose f is analytic on $\mathbb{D}$, and, in fact, maps $\mathbb{D}$ into $\mathbb{D}$, with $f(0) = 0$, $f'(0) = a$, and $|a| < 1$. Let M_a denote the Möbius transformation

$$M_a(w) = \frac{w + a}{1 + \overline{a}w}.$$

Show there exists a function $h(z)$, analytic on $\mathbb{D}$, with $|h(z)| \leq 1$ for all $z \in \mathbb{D}$ and $h(0) = 0$, such that

$$f(z) = zM_a(h(z)) \text{ for all } z \in \mathbb{D}.$$

8. Suppose f is analytic and bounded in $\mathbb{D}$ and let $M = \sup_{z \in \mathbb{D}} |f(z)|$. Suppose further that $f(0) = 1$. Let $n(r)$ denote the number of zeros of f in the disc $\{z : |z| < r\}$, a zero of multiplicity m being counted m times. Show

$$n(r) \leq \frac{\ln M}{\ln(1/r)}.$$

9. Suppose u is a non-constant harmonic function on the region G in $\mathbb{C}$. Let

$$G_0 = \{z \in G : \frac{\partial u}{\partial x}(z) = \frac{\partial u}{\partial y}(z) = 0\}.$$

Show G_0 has no points of accumulation in G, i. e., for every $z_0 \in G$, there exists $\epsilon > 0$, such that $0 < |z - z_0| < \epsilon$ implies z is not in G.

10. Let f be analytic in a region G and for $w \in \mathbb{C}$, let $\mu(w)$ be the number of zeros in G of $f(z) - w$, counted according to multiplicity. Suppose $\mu(w_0) \geq n$, where $w_0 \in \mathbb{C}$ and n is a positive integer. Show there exists a neighborhood W of w_0 such that every $w \in W - \{w_0\}$ has at least n distinct pre-images under f in G.

11. Given f_n analytic, mapping $\mathbb{D}$ in $\mathbb{D}$, $n = 1, 2, 3, ...$ Let $A = \{\frac{1}{k} : k = 2, 3, 4, ...\} \cup \{0\}$. Suppose for every $a \in A$, that $\lim_{n \to \infty} f_n(a)$ exists in $\mathbb{C}$. Show there exists a complex-valued function f on $\mathbb{D}$ such that $\lim_{n \to \infty} f_n(z) = f(z)$ for all $z \in \mathbb{D}$, the convergence being uniform on every compact subset of $\mathbb{D}$.