Qualifying Exam in Complex Analysis, May 2001

May 18, 2001

Instructions: Do eight problems. State clearly any results you use.

1. (a) Give an example of two simply connected domains $\Omega$ and $\Omega'$ in $\mathbb{C}$ which are real $C^\infty$ diffeomorphic, but are not biholomorphically equivalent. Explain.
(b) Construct a biholomorphic equivalence between the unit disk, $D$, and the upper half plane, $\mathbb{H}$.

2. Let $u$ be harmonic on a connected domain $\Omega$ in $\mathbb{C}$
   (a) Show $f = u_x - iu_y$ is holomorphic in $\Omega$.
   (b) Suppose $u$ is the real part of a holomorphic function $g$ in $\Omega$. Show $g' = f$.
   (c) Give a necessary and sufficient condition on $\Omega$ so that every harmonic function is the real part of a holomorphic function. Justify your answer.

3. Calculate $\int_{\infty}^{\infty} \frac{e^{-z^2}}{(1+z^2)^2} \, dz$ using the residue theorem. Justify all the steps in your calculation.

4. (a) Let $\Omega$ be a connected domain in $\mathbb{C} = \mathbb{R}^2$ and $f$ a smooth local diffeomorphism $f : \Omega \to \mathbb{R}^2$. What does it mean to say $f$ is conformal?. Give the geometric definition.
   (b) Show that $f : \Omega \to \mathbb{C}$ is conformal if and only $f$ is holomorphic and never vanishes on $\Omega$.

5. (a) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a domain, $\Omega$. Let $\Omega^- = \{ z \in \mathbb{C} : z^- \in \Omega \}$. Define $f^* : \Omega^- \to \mathbb{C}$ by $f^*(z) = f(z^-)^{-1}$. Show $f^*$ is holomorphic on $\Omega^-$. 
   (b) Suppose $\Omega$ is connected and non-empty and $\Omega^- = \Omega$. Show $\Omega \cap \mathbb{R}$ is non-empty, and, if $f$ is holomorphic in $\Omega$, $f^* = f^*$ if and only if $f(x)$ is real for all $x \in \Omega \cap \mathbb{R}$.

6. Let $f$ be a non-constant holomorphic function on $\Omega$, where $\Omega$ is a bounded connected domain and $|f|$ is constant on $\partial(\Omega)$. Show $f$ must have a zero in $\Omega$. (Assume $f$ is co-nt. on $\partial(\Omega)$)
7. (a) State Schwartz's lemma.
   (b) Let $Aut(D)$ be the group of holomorphic automorphisms of the unit disk $D$. Find $Aut(D)$

8. Let $u$ be a harmonic function on a connected domain $\Omega$ containing a disk, $D$ which is identically zero on the boundary $\partial D$. Show $u$ is identically zero on $D$. Show if two harmonic functions agree on $\partial D$, then they must coincide on $\Omega$.

9. Let $D(a, r_0)$ denote the open disk $\{z : |z - a| < r_0\}$.
   (a) Suppose $f$ is holomorphic in $D(a, r_0)$. Show that (*) there exists a non-decreasing function $M : (0, r_0) \to (0, \infty)$ such that $|f^{(n)}(a)| \leq \frac{M(n)}{r^n}$ for all integers $n \geq 0$ and $r \in (0, r_0)$.
   (b) Suppose $f$ is holomorphic in $D(a, r_1)$ for some $r_1 \in (0, r_0)$ and satisfies (*) above. Show $f$ extends holomorphically to $D(a, r_0)$.

10. Let $D = \{z : |z| < 1\}$. Suppose $f_n$ is a sequence of holomorphic functions on $D$ and $\exp(f_n(z)) \to g(z)$ uniformly on compacta of $D$. If $g(0) = 0$, what can be said about $g$? Explain.

11. For a domain $\Omega$ in $\mathbb{C}$, a subset $A$ of $\Omega$ is said to be locally finite in $\Omega$ if $A \cap K$ is finite for each compact set $K$ in $\Omega$. Suppose $A$ is locally finite in $\Omega$ and $f_n$ is a sequence of holomorphic functions on $\Omega$ which converges uniformly on compacta on $\Omega - A$.
   (a) Show $\Omega - A$ is open.
   (b) Show there is a unique holomorphic function $f$ on $\Omega$ such that $f_n$ converges uniformly on compacta to $f$ on $\Omega$.

12. Suppose $\Omega$ is a simply connected domain and $g$ is a holomorphic function in it which is not identically zero. Let $n > 1$ be an integer. Show $g$ has a holomorphic $n^{th}$ root on $\Omega$ if and only if every zero of $g$ in $\Omega$ has multiplicity divisible by $n$.

13. Suppose $f$ is holomorphic on an open set containing $D^-$, where $D$ is the open disk $\{z : |z| < 1\}$ and that $f$ has at least 2 zeros (or one zero with multiplicity at least 2) in $\{z : |z| \leq \frac{1}{2}\}$. Show $|f(0)| \leq \frac{1}{4}$.

14. For $k = 0, 1, \ldots$ show that $\int_0^\infty te^{-zt}dt$ converges in the right half plane, $R = \{z : \Re z > 0\}$ to a holomorphic function $F_k(z)$ and in fact $F_k(z) = \frac{H_k}{z^{1+R}}$ for $z \in R$.
   Hint: Show $F_{k+1}(z) = -zF_k(z)$ for all $k = 0, 1, \ldots$ and $z \in R$. 