Differential Geometry
Fall 2012

Do any 6 problems. *Note:* Throughout this exam, all manifolds are $\mathcal{C}^\infty$ and connected, and all maps are $\mathcal{C}^\infty$ are unless it is specifically stated otherwise.

1. Let M be a manifold, and let U and V be two open subsets such that $M = U \cup V$ and $U \cap V$ is connected. Show that if ω is a smooth 1-form on M such that $\omega|U$ and $\omega|V$ are exact, then ω is exact.

2. (a) Use the result of the preceding problem to show that the first de Rham cohomology space $H^1_{dR}(\mathbb{S}^n)$ vanishes.

   (b) Exhibit an explicit basis of the first de Rham cohomology $H^1_{dR}(\mathbb{T}^2)$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

3. Let $\omega = \cos z dx \wedge dy + x \sin z dy \wedge dz + y dy \wedge dz$. Compute

$$\int_{\mathbb{S}^2} \omega,$$

   where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere centered at the origin with the orientation given by the exterior unit normal vector. State clearly any theorem that you use to do the calculation.

4. Suppose $a(z)$, $b(z)$ are two real-valued C^1 functions on an open interval I. Consider two vector fields on $\mathbb{R}^2 \times I$, $Z_1 = \frac{\partial}{\partial z}$ and $Z_2 = a(z)\frac{\partial}{\partial x} + b(z)\frac{\partial}{\partial y}$. Assume further that the functions $a(z)$ and $b(z)$ do not vanish on I. Prove that the distribution spanned by Z_1, Z_2 is integrable if and only if there exists a constant C such that $a(z) = Cb(z)$.

5. Let M be a smooth manifold of two dimensions. Let ω be a nowhere vanishing exterior form of degree two. Show that every point of M lies in a coordinate system (U, x, y) such that $\omega|U = dx \wedge dy$.

6. Show that the matrix $\begin{bmatrix} -3 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \in SL(2, \mathbb{R})$ is not in the image of $\exp : sl(2, \mathbb{R}) \to SL(2, \mathbb{R})$.

   Hint: It might help to compute the matrix exponential explicitly, which simplifies in terms of the determinant:

$$\exp\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = \frac{1}{d}\begin{bmatrix} d\cosh(d) + a\sinh(d) & b\sinh(d) \\ c\sinh(d) & d\cosh(d) - a\sinh(d) \end{bmatrix}$$

   where $d = \sqrt{a^2 + bc}$ (and may be complex).

7. Let M be a complete Riemannian manifold and $\gamma : (-1,1) \to M$ a smooth curve. Suppose X_s is a vector field along γ such that $|X_s| \equiv 1$ and $\langle X_s, \dot{\gamma}(s) \rangle \equiv 0$. Consider the mapping $E : (-1,1) \times (-\infty, \infty) \to M$ given by

$$E(s,t) = \exp_{\gamma(s)}(tX_s).$$

Show that for every (s_0, t_0) the curve $s \to E(s, t_0)$ is perpendicular to the geodesic $t \to E(s_0, t)$ at (s_0, t_0).

8. Prove that the group $GL^+(n, \mathbb{R})$ of invertible matrices with positive determinant is connected for all $n \geq 1$.

9. Let S be a surface of revolution in $\mathbb{R}^3$ equipped with the induced metric. Show that the curves of intersection of S with planes containing the axis of revolution are geodesics on S.

10. Suppose X and Y are two Riemannian manifolds and $f : X \longrightarrow Y$ is a diffeomorhpism satisfying $< df_p v, df_p w >= \lambda(p) < v, w >$ for every point $p \in X$ and every pair of tangent vectors $v, w \in T_pM$ where $\lambda : X \longrightarrow \mathbb{R}$ is a *bounded* function. Prove that Y is geodesically complete if and only if X is geodesically complete.