Differential Geometry
Spring 2010

Do any 6 problems. *Note:* Throughout this exam, all manifolds are C^∞ and connected, and all maps C^∞ are unless it is specifically stated otherwise.

1. Carefully state the theorem about existence of partitions of unity on a smooth manifold. Use this theorem to prove (without invoking the notion of a Riemannian metric) that if X is a smooth, nowhere vanishing vector field on a manifold M then there exists a differential form ω on M satisfying $\omega(X) \equiv 1$.

2. Let $S \subset \mathbb{R}^2$ be the unit square made up of closed line segments joining $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$. Prove that S is not a smooth submanifold of $\mathbb{R}^2$.

3. Consider the metric $ds^2 = dx^2 + e^{2x}dy^2$ on $\mathbb{R}^2$. Show that the curves $y = \text{const}$ are geodesics. Conclude that the vector field $\partial/\partial y$ is a Jacobi vector field along $y = \text{const}$ and calculate the Gaussian curvature of this metric. *Hint:* Apply the Jacobi equation to the variation generated by the $y = \text{const}$ curves or else solve by direct computation.

4. Let S be a surface of revolution in $\mathbb{R}^3$ equipped with the induced metric. Show that the curves of intersection of S with planes containing the axis of revolution are geodesics on S. *Hint:* No calculations are necessary to do this problem.

5. Prove that the group $SO(n, \mathbb{R})$ is connected for all $n \geq 1$.

6. Let k, l be positive integers and $n = k + l$. Consider $G \subset Sl(n, \mathbb{R})$ consisting of those matrices $A = (a_{ij})$ whose lower left-hand $k \times l$ block has all entries equal to zero, i.e. $A \in G$ if and only if $a_{ij} = 0$ for $i \leq k$, $j \leq l$. Prove that G is a Lie subgroup of $Gl(n, \mathbb{R})$ and determine the Lie algebra of G.

7. Let S be a surface equipped with a complete Riemannian metric and $\gamma : (-1,1) \to S$ a smooth curve. Suppose X_t is a parallel vector field along γ such that $|X_t| \equiv 1$ and $\langle X_t, \dot\gamma(t)\rangle \equiv 0$. Consider the mapping $E : (-1,1) \times (-\infty, \infty) \to S$ given by
$$E(s,t) = \exp_{\gamma(s)}(tX_s).$$
Show that the curves $s \to E(s, t_0)$ are perpendicular to geodesics $t \to E(s_0, t)$ for all (s_0, t_0).

8. Let $N = (0, \ldots, 0, 1)$ and $S = -N$ be the "north pole" and "south pole" in $\mathbb{S}^n$ respectively. Define stereographic projection $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ regarding N by
$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

Show that

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{1 + |u|^2}$$

and compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$. Verify that the atlas consisting of charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$.

9. Let D be a geodesic quadrilateral in the hyperbolic plane with three right angles and the fourth angle equal to α. Derive a formula for the area of D. What are all possible values of α?

10. A curve $\gamma : [0, \infty) \longrightarrow M$ in a Riemannian manifold M is called *divergent* if and only if γ escapes from every compact set K (that is, if for every compact set $K \subset M$, there exists $t_0 > 0$ such that $\gamma(t) \in M \setminus K$ for $t > t_0$). Prove that M is complete if and only if

$$\lim_{t \to \infty} \int_0^t |\gamma'(u)| \, du = \infty$$

for every piecewise smooth, divergent curve in M.