# Differential Geometry Qualifying Exam
# Spring 2011

**Instructions:** No more than five problems will be graded—specify which ones you want graded.

**Problem 1.** Let $(M, g)$ and $(M', g')$ be $n$-dimensional Riemannian manifolds.

(a) Define what it means for $(M, g)$ and $(M', g')$ to be *isometric.*

(b) Define what it means for $(M, g)$ and $(M', g')$ to be *conformally equaivalent.*

(c) Let

$$M = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, z > 0\}$$

and

$$M' = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = z^2, z > 0\},$$

each with metric induced by the standard Euclidean metric in $\mathbb{R}^3$. Prove or disprove: $M$ and $M'$ are isometric.

**Problem 2.** Give a precise definition of a *smooth vector bundle.*

(a) Define what it means for a smooth vector bundle to be trivial.

(b) Give an example of a smooth vector bundle that is not trivial and prove that it is not trivial.

**Problem 3.** Prove that the set $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 = y^3\}$ is not a smooth submanifold of $\mathbb{R}^2$.

**Problem 4.** Let $M$ be a smooth hypersurface in $\mathbb{R}^n$. Prove that $M$ is orientable if and only if there exists a transverse smooth vector field along $M$. (For this problem, you may use any intrinsic definition of orientability.)

**Problem 5.** Consider vector fields on $\{(x, y, z) \in \mathbb{R}^3 : x, y, z > 0\}$ given by

$$X = xy\frac{\partial}{\partial x} - yz\frac{\partial}{\partial z} \text{ and } Y = x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y}.$$

(a) Let $\phi_t$ and $\psi_s$ be the flows generated by $X$ and $Y$ respectively and compute $\phi_t(x_0, y_0, z_0)$ and $\psi_s(x_0, y_0, z_0)$.

(b) Do $X$ and $Y$ span an involutive distribution? Explain.

**Problem 6.** Show that the smooth manifold $S^1 \times S^2$ does not admit a metric whose sectional curvatures are all positive.



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**Problem 7.** Let $\Sigma$ be a smooth surface of genus equal to 2.

(a) Draw the image of an embedding $\phi : \Sigma \to \mathbb{R}^3$. Label at least one point of positive and negative curvature, respectively.

(b) Prove that for any embedding $\phi$ there is an open subset of $\phi(\Sigma)$ where curvature is negative.

**Problem 8.** Consider the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x = y^2 + z^2 \leq 4\}$$

oriented in such a way that the basis $\left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial y} - \frac{\partial}{\partial z} \right\}$ of $T_{(0,0,0)}M$ is positively oriented. Compute:

(a) $\displaystyle\int_M xy^2\, dx \wedge dz + \sin(yz^2)\, dy \wedge dz$

(b) $\displaystyle\int_{\partial M} xz\, dy + y^2\, dz.$

**Problem 9.** Let $G$ be a connected Lie group with bi-invariant metric $g$. Let $\nabla$ denote the Levi-Cevita connection. Then for left invariant vector fields $X, Y$ and $Z$,

$$g([X,Y], Z) = g(X, [Y, Z]) \text{ and } \nabla_X Y = \frac{1}{2}[X, Y].$$

(a) Show that the curvature tensor of $\nabla$ satisfies

$$Rm(X, Y, Z, W) = -\frac{1}{4} g([X, Y], [Z, W])$$

whenever $X, Y, Z$ and $W$ are left invariant vector fields.

(b) Show that the sectional curvature of $(G, g)$ is non-negative.

(c) Prove that the Lie algebra $\mathfrak{g}$ of $G$ has zero bracket if and only if the sectional curvature of $(G, g)$ is zero.