QUALIFYING EXAMINATION

DIFFERENTIAL GEOMETRY

September 13, 2004

A C^∞ manifold is a Hausdorff topological space with a countable basis which is equipped with a C^∞ structure.

Answer any six questions.

1. Prove that the set of 2×2 matrices of rank one is a three-dimensional submanifold of the space $M(2, \mathbb{R})$ of all 2×2 matrices with real entries.

2. Prove that every paracompact C^∞ manifold carries a C^∞ Riemannian metric.

3. Consider the unit 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Let x,y and z be the restrictions of the coordinate functions of $\mathbb{R}^3$ to $\mathbb{S}^2$. Prove that
$$dx = \frac{-ydy - zdz}{x}$$
and compute $\int_\gamma (dx + dy + dz)$ where γ is a curve connecting (0,1,0)to $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

4. Let M be a C^∞ manifold and $f : M \to \mathbb{R}$ a smooth function.
   (a) Suppose $df_x \neq 0$ at every point where $f(x) = 0$. Prove that $N = f^-1(0)$ is a smooth submanifold of M of codimension one.
   (b) Prove that if M and f are as above and M is orientable, then N is orientable. *Hint:* Use a Riemannian metric if you like.

5. Show that that the real projective plane $\mathbb{RP}^2$ is a submanifold of $\mathbb{R}^4$. *Hint:* Let $f : \mathbb{S}^2 \to \mathbb{R}^4$ be the map defined by
$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$
Show that f factors through a mapping $g : \mathbb{RP}^2 \to \mathbb{R}^4$. Then show that g is an embedding.

6. Let $f : M \to N$ be a C^∞ map between C^∞ manifolds, where M is compact. Define a regular value of f. Show that the cardinality of the set $\{f^{-1}(y)\}$ where y ranges over the regular values of f defines a locally constant finite function.

7. Find the geodesics of

   (a) the n-sphere $\mathbb{S}^n$.

   (b) the right circular cylinder $\{ (x, y, z) \mid x^2 + y^2 = 1, \ z \in \mathbb{R} \}$.

   Explain.

8. Prove Gauss' Lemma that states that in the geodesic ball around a point p in a Riemannian manifold, the geodesic segments through p are the orthogonal trajectories of the hypersurfaces $\{ \exp_p(v) \mid |v| = \text{const} \}$.

9. By using polar coordinates on the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, show that $\mathbb{S}^2$ is orientable.

10. Suppose that $a(z)$, $b(z)$ are real $\mathbb{C}^1$ functions defined on an interval I. Let x, y denote the standard coordinates on $\mathbb{R}^2$. Consider the vector fields on $\mathbb{R}^2 \times I$, $z_1 = \frac{\partial}{\partial z}$ and $z_2 = a(z)\frac{\partial}{\partial x} + b(z)\frac{\partial}{\partial y}$. Prove that the distribution spanned by z_1 and z_2 is integrable if and only if there exists a real constant A such that $a = Ab$.