Qualifying Exam in Differential Geometry, Spring 2003

May 15, 2003

Instructions: For those taking the Differential Geometry Qualifying Exam do any 7 of the following problems. Those taking the Final Exam for the Differential Geometry course, Spring 2003 do any 4 of the following problems. Please write Qualifying Exam or Final at the top of your paper. In answering questions always explain the reasons for any statements you make in your solution. If you can do most, but not all, of a problem do it and don’t worry.

1. Let $V$ be a real vector space of dimension $n$ and $1 \leq r \leq n$.
   (a) Define the real Grassmann manifold, $G^r(V)$, of all $r$ dimensional subspaces of $V$ and explain how one puts a manifold structure on it.
   (b) Prove $G^r(V)$ is compact and connected. Calculate its dimension.

2. (a) Calculate the Lie algebra, $g$, of the Lorentz group, $SO(n,1)$.
   (b) Defining hyperbolic space, $H^n$, as $SO(n,1)/SO(n)$. Explain why the isometry group of $H^n$ acts transitively.
   (c) In $g$ identify the tangent space to the identity of $H^n$.

3. Let $X$ be a compact, connected, orientable smooth R-surface and $K$ be its Gaussian curvature.
   (a) Show if $g = 0$ then there must be a point $p \in X$ where $K(p) > 0$.
   (b) Show if $g \geq 2$ then there must be a point $p \in X$ where $K(p) < 0$.
   (c) Show if $g = 1$, then no conclusion can be drawn by giving an example of a R-surface with $g = 1$ where $K$ is identically zero and another example of a R-surface with $g = 1$ where $K$ takes
positive, negative and zero values. Sketch a picture of this last one.

4. Let $C$ be a smooth curve in the $x, z$ plane in $\mathbb{R}^3$ and rotate it about the $z$ axis to create a smooth surface of revolution, $X$.

(a) Prove that every plane section of $X$ through the $z$ axis is a geodesic of $X$.
(b) Now consider the sections of $X$ formed by horizontal planes. Which of these are geodesics? Prove your answer.

5. (a) Write the Euler-LaGrange equations for a geodesic on a surface. Calculate the geodesics on the Poincare upper half plane, $H^+$.
(b) Explain why the figure bounded by the lines $x = \pm 1/2$ and the unit circle is a geodesic triangle in $H^+$.
(c) Calculate its hyperbolic angles and its area.
(d) Let $p_+ = (\frac{1}{2}, \sqrt{3})$ and $p_- = (-\frac{1}{2}, \sqrt{3})$ be points in $H^+$. Find the unique geodesic joining them. What is its length?
(e) Consider the Euclidean straighth line joining $p_+$ and $p_-$. What is its (hyperbolic) length?

6. Let $(X, g)$ be an $n$-dimensional R-manifold.

(a) Define an isometry of $(X, g)$.
(b) Find the volume form on $X$, Explain.
(c) Show each isometry preserves volume.
(d) What is the volume form on the hyperbolic plane. Find an open region in $H^+$ which is unbounded, but has finite area, by calculating the area using the volume form.

7. Let $X$ be a compact connected orientable smooth R-surface and $w$ be a tangential vector field on $X$ with only isolated singularities.

(a) Explain why the number of singularities is finite.
(b) Define the index of such a singularity. Explain why it's an integer.
(c) Draw a picture of a tangential vector field together with its integral curves having an isolated singularity of index -2.
(d) Draw a picture of a tangential vector field together with its integral curves having an isolated singularity of index 3.

(e) State Poincaré’s index theorem.

(f) Explain why on the two sphere $S^2$ every tangential vector field must have a singularity. State a generalization of this statement to higher dimensions.

(g) Explain why on a surface of genus $g \geq 2$ every tangential vector field must have a singularity.

(h) Define parallelizable. Explain why any manifold diffeomorphic with the two torus, $T^2$, is parallelizable.

8. (a) Consider the vector valued differential equation\[ \frac{d\mathbf{v}}{dt} = A(t)\mathbf{v}(t) + b(t), \quad \mathbf{v}(0) = \mathbf{v}_0, \] where $\mathbf{v}(t)$ is the unknown vector valued solution, $A(t)$ is the given real matrix of coefficients and $b(t)$ a given vector valued function. Explain why such an equation always has a unique global solution.

(b) Solve this equation explicitly when $A(t) = A$, a constant and $b(t) = 0$.

(c) Let $\gamma$ be a smooth curve in $\mathbb{R}^n$ and \{e_1(t), \ldots, e_n(t)\} an orthonormal frame along the curve with the base point of the curve as the origin (i.e. a curve in the real Stiefel manifold of $n$-frames). Then $\gamma'(t) = A(t)\gamma(t)$ for some matrix function of $t$. Prove $A(t)$ is skew symmetric for all $t$.

9. Using the change of variable formula do the following:

(a) Let $T$ be positive definite self adjoint operator on $\mathbb{R}^n$ and $dx$ Lebesgue measure. Show
\[ \int_{\mathbb{R}^n} e^{-(Tx,x)} dx = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det T}}. \]

(b) Prove that the hyperbolic area in $H^+$ is invariant under $SL(2,\mathbb{R})$.

10. Let $X$ be a simply connected $R$-manifold of non-positive sectional curvature at every point and $p$ and $q$ be any two distinct points.

(a) Prove that there is at most one geodesic joining $p$ and $q$. (If you are not comfortable with the case of an $n$ manifold let $X$ be a 2 manifold).
(b) Explain why statement a. is false if $X$ is not simply connected.

(c) Explain why statement a. is false if the curvature assumption fails to hold.

(d) Explain under what circumstances it could happen that no geodesic exists joining $p$ and $q$.

(e) Explain why two different geodesics emanating from $p$ cannot have angle zero between them.

(f) Can you guess the statement of a result that says a simply connected $R$-manifold of non-positive sectional curvature at every point is diffeomorphic with $\mathbb{R}^n$.

11. Let $X$ be a R-2 manifold.

(a) Define what is meant by a system of local geodesic polar coordinates $U(r, \theta)$ about a point $p$.

(b) Explain why in such an $U(r, \theta)$ the curves where $r$ is constant meet those where $\theta$ is constant orthogonally.

(c) State Jacobi’s differential equation.

(d) Prove if $q = (r, \theta) \in U$ and $\gamma$ is any smooth curve in $U$ joining $p$ and $q$, then $r \leq L(\gamma)$ with equality if and only if $\gamma$ is the radial geodesic joining $p$ and $q$.

12. Let $X$ be a R-2 manifold and $U(r, \theta)$ a system of local geodesic polar coordinates about a point $p$, where $K(p) \neq 0$.

(a) Explain why for small $r > 0$ and fixed $\theta$, $g(r, \theta) = r - \frac{r^3}{6} K(p) +$ higher powers of $r$.

(b) For fixed small $r > 0$, let $\gamma_r$ be the curve where $r$ is constant and $L(r)$ its length. Prove that $L(r) = 2\pi r - \frac{3}{2} K(p) r^3 +$ higher order terms. So if $K(p) > 0$ then locally $L(r)$ first increases with increasing $r$ and then decreases, while if $K(p) < 0$ then locally $L(r)$ just increases.

(c) Prove $K(p) = \frac{2}{3} \lim_{r \to 0} \frac{2\pi r - L(r)}{r^3}$.

(d) Let $A(r)$ be area enclosed by $\gamma_r$. Show $A(r) = \pi r^2 - \frac{\pi}{12} K(p) r^4 +$ higher order terms and hence

(c) Prove $K(p) = \frac{12}{\pi} \lim_{r \to 0} \frac{\pi r^2 - A(r)}{r^4}$. 

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