

Logic Comprehensive Exam

Fall 2010

Answer four questions from each part.

Part One

- Express the following sentences in first order logic, and for each sentence note its quantifier rank. (You should make the quantifier rank as small as possible, but you need not prove that you have done so.)
 - In the language with $=$, but no other nonlogical symbols: “there are exactly p different elements.”
 - In the language of ordered rings $\{+, \times, <, 0, 1\}$: “there are infinitely many twin primes.”
 - In the language of set theory $\{\in\}$: “ x is the power set of y ” (i.e. x is the set of all subsets of y).
 - In the language of graphs $\{E\}$: “the degree of each vertex is at most 3.”
 - For a subset $A \subseteq \mathbb{R}$, the sentence in the language in the language $\{<, \in\}$ and the set parameter A (i.e. expressions of the form $x \in A$ are allowed): “ x is in the topological closure of A .”
- Let F be an infinite field of characteristic p . Show that F is elementarily equivalent to a field F' of characteristic p such that F' contains an element transcendental over its prime subfield. (The prime subfield is the subfield generated by the multiplicative identity.)
- Let F be a filter on I . Prove that F is an ultrafilter iff F is maximal.
- Assume that every model of a sentence φ satisfies a sentence from Σ . Show that there is a finite $\Delta \subseteq \Sigma$ such that every model of φ satisfies a sentence in Δ .
- Prove that in a finite language which contains a unary function symbol f , the Zero-One Law cannot hold. Suggestion: consider the formula $\forall x(f(x) \neq x)$.
- Prove that if a theory has up to isomorphism only countably many countable models, then it has a countable ω -saturated model.

7. Let \mathcal{F} be the set of formulas of propositional logic containing only \wedge , \vee and \neg . Define a mapping $*$ on the set \mathcal{F} as follows:

- (a) if P is atomic, $P^* = \neg P$;
- (b) $(A \wedge B)^* = A^* \vee B^*$;
- (c) $(A \vee B)^* = A^* \wedge B^*$;
- (d) $(\neg A)^* = \neg A^*$.

Prove, for any formula $X \in \mathcal{F}$, $X^* \equiv \neg X$ is a tautology.

8. Let \mathcal{L} be a finite or countable language, and T a complete theory in \mathcal{L} . Prove that a model \mathcal{A} of T is atomic if and only if \mathcal{A} is a prime model of T . (Recall that a model of T is *atomic* if the only types it realizes are principal types over T , and *prime* if it is elementarily embeddable into every model of T .)

Part Two

1. Let A be any computable subset of $\text{Th}(\mathcal{N})$, the set of sentences in the language of arithmetic which hold of the natural numbers. Let $\text{Con}(A)$ be the sentence saying that there is no proof of the sentence $0 \neq 0$ from the axiom set A . Use the Second Incompleteness Theorem to show that there exists a structure \mathcal{M} which is a model of $A \cup \{\neg \text{Con}(A)\}$. (Thus \mathcal{M} is a model of A , yet believes that A is inconsistent.)
2. $\mathcal{N}_S = (\omega, 0, S)$ is the structure of the natural numbers under the successor function. Fix any decidable axiom set A_S for \mathcal{N}_S such that every model of A_S consists of a copy of \mathcal{N}_S along with an arbitrary number of “ \mathbb{Z} -chains.” Prove that the theory of \mathcal{N}_S is decidable. (Suggestion: apply Vaught’s Test to $\text{Cn}(A_S)$, the set of all consequences of A_S .)
3. Assume that $\Gamma \vdash \phi$, and that P is a relation symbol which does not occur in ϕ nor in any formula of Γ . Prove that there exists a deduction of ϕ from Γ such that P does not appear in any formula of the deduction.
4. Let \mathcal{C} be a set of sets. \mathcal{C} is said to be of *finite character* provided: $S \in \mathcal{C}$ iff $F \in \mathcal{C}$ for every finite $F \subseteq S$. Tukey’s Lemma says that every set of finite character has a maximal element. Show Zorn’s Lemma implies Tukey’s Lemma.

5. In the context of ZFC set theory, define cardinal number, and prove that every set is equinumerous with exactly one cardinal number.
6. Let W_e denote the c.e. set which is the domain of the e -th Turing program. (This is the standard indexing of c.e. sets.) At which level of the arithmetic hierarchy does each of the following sets appear? To justify your answers, write out an exact definition of each set, using as few quantifiers as possible.
 - (a) $A_1 = \{e : W_e \text{ contains no even numbers}\}$.
 - (b) $A_2 = \{e : W_e \text{ contains some element of } K\}$.
 - (c) $A_3 = \{e : W_e \text{ contains every element of } K\}$.
 - (d) $A_4 = \{e : W_e \text{ is infinite}\}$.

Here K represents the Halting Problem, or any other Σ_1^0 -complete set you prefer.

7. Let $A \subseteq \omega$ be an infinite set. Write out an explicit oracle Turing program such that with A as oracle, the program computes the function

$$f(x) = \mu y(y > x \text{ and } y \in A).$$

Next, show that there is no oracle Turing program which, given any (finite or infinite) oracle B , computes the function

$$g(x) = \begin{cases} \mu y(y > x \text{ and } y \in B) & \text{if there exists such a } y \\ 0 & \text{if not.} \end{cases}$$

8. Show that the set $\text{Fin} = \{e : W_e \text{ is finite}\}$ is a Σ_2 -complete set with respect to 1-reducibility \leq_1 . (The hard part is showing completeness, but be sure also to show that Fin is Σ_2 .)