Logic Comprehensive Exam
Spring 2001

Answer four questions from each part.

Part One

1. Let $(A, <)$ and $(B, <)$ be two dense linear orderings without end points, such that $A \subseteq B$. Prove that $(A, <)$ is an elementary submodel of $(B, <)$.

2. Let L be the first-order language of group theory, and let S be a sentence of L. Suppose there are arbitrarily large finite groups in which S is true. Show there is a countable group in which S is true.

3. We use the following notation for substitution in first-order logic. $\Phi[x/t]$ is the formula that is like Φ except that occurrences of the term t have been substituted for free occurrences of the variable x. Now, let S be a set of closed formulas (sentences) of first-order logic, let $(\exists x)\Phi$ be a closed formula, and let c be a constant symbol that does not occur in Φ or in any member of S. Show that if S is satisfiable, so is $S \cup \{(\exists x)\Phi \supset \Phi[x/c]\}$.

4. Let F be an infinite set of propositional formulas and suppose, for each Boolean valuation v, at least one member of F maps to true under v. Show there is a finite subset $\{X_1, \ldots, X_n\} \subset F$ such that $(X_1 \vee \ldots \vee X_n)$ is a tautology. Hint: consider $G = \{\neg X \mid X \in F\}$.

5. Suppose that $\mathcal{A}_0 \prec \mathcal{A}_1 \prec \cdots \prec \mathcal{A}_n \prec \cdots$ is a chain of elementary extensions, and let $\mathcal{A} = \cup_{i=0}^{\infty} \mathcal{A}_i$. Prove that $\mathcal{A}_0 \prec \mathcal{A}$.

6. Prove the Compactness Theorem for first-order logic using ultrapowers.

7. This problem is about finite axiomatizability.

   (a) Suppose that φ_0, φ_1, ... is an infinite sequence of sentences such that, for all m, n, if $m < n$ then $\varphi_n \supset \varphi_m$ and $\neg(\varphi_m \supset \varphi_n)$ are theorems of first order logic. Show that for every sentence ψ, $\mathrm{CN}\{\varphi_0, \varphi_1, \ldots\} \neq \mathrm{CN}\{\Psi\}$, where CN$S$ denotes the set of logical consequences of S.

   (b) Show that the theory of fields of characteristic 0 is not finitely axiomatizable.

8. Let D be an ultrafilter on $\mathbb{N}$, and let $d : \mathfrak{A} \to \prod_D \mathfrak{A}$ be the natural embedding of $\mathfrak{A}$ into its ultraproduct, defined by $d(a) = \langle a \mid n \in \mathbb{N} \rangle$. Knowing that d is an isomorphism, what can you conclude about $\mathfrak{A}$ and D? Provide a proof.

Part Two

1. Sketch a proof of Tarski's Theorem: the truth set of arithmetic is not arithmetically definable. If you prefer, you may show the set of sentences in the language of set theory that are true in the model of hereditarily finite sets is not representable in this model, by any formula.

2. Prove that if A is recursively enumerable and f is a partial recursive function, then $f(A)$ and $f^{-1}(A)$ are recursively enumerable. Since there are several ways recursive enumerability can be defined, begin by giving the version you will use.

   Notation: $f(A) = \{f(a) \mid a \in A\}$ and $f^{-1}(A) = \{a \mid f(a) \in A\}$.

3. Goldbach's famous, as yet unproven and unrefuted, conjecture says that every even integer ≥ 4 can be written as the sum of two primes. Show that if (the natural formalization of) this statement is undecidable in Peano Arithmetic then Goldbach's conjecture is true.

4. Call a theory $\mathcal{T}$ ω-incomplete if there is a formula $A(x)$ such that $\mathcal{T} \vdash A(t)$ for each closed term t, but not $\vdash (\forall x)A(x)$. Prove that ω-inconsistency implies ω-incompleteness for consistent theories.

5. We use R_n to denote the recursively enumerable set with index n. Let $S = \{n \mid 3 \in R_n\}$. Show that S is recursively enumerable but not recursive.

6. A Δ_0 formula of arithmetic is a formula all of whose quantifiers are bounded. A Σ_1 formula is a formula of the form $(\exists x)\varphi$ where φ is a Δ_0 formula. A theory is Δ_0 *complete* if every true (in the standard model) closed Δ_0 formula is provable in it. Similarly for Σ_1 complete. The theory PA is Δ_0 complete. Is it Σ_1 complete? Give reasons.

7. Sketch a proof of Gödel's First Incompleteness Theorem using Tarski's Theorem.

8. Let F be the set of formulas of propositional logic with only $\supset$ as connective. Assume formulas are identified with particular hereditarily finite sets (members of R_ω), and the class of formulas is represented by the Σ formula $F(x)$. Take as axioms all formulas of the forms: $P \supset (Q \supset P)$ and $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$, and take modus ponens as the only rule of derivation. Show the set of theorems of this axiom system is also Σ.