Logic Comprehensive Exam/Final Exam for MATH 71200

In Three Parts

Spring 2011

If you only want to take the Final Exam for MATH 7200, then you only need to work on part two of this exam.

Part Zero

Choose two of the following four theorems. Give a precise formulation, and sketch the proof. Not all details must be included.

1. The Tarski-Vaught Test for elementary substructures.
2. The (downward) Löwenheim-Skolem Theorem.
3. Either the Finiteness Theorem (a.k.a. Compactness Theorem), or the Completeness Theorem.
4. Either the First or the Second Incompleteness Theorem.

Do both of the following problems.

1. In a first-order language L with logical connectives $\wedge$, $\vee$, $\neg$ and the quantifier $\exists$, let Σ be a set of L-sentences such that every simply primitive L-formula is equivalent modulo Σ to a quantifier-free formula. Prove, by induction on the complexity of formulas in L, that Σ has quantifier elimination.

   Remark: As a reminder, a formula is simply primitive if it is of the form $\exists x\psi$, where ψ is a conjunction of literals (i.e., atomic formulas and negated atomic formulas).

2. Express the following statements in first order logic.
   (a) In the language with $=$, but no nonlogical symbols, for a positive integer p: "there are at most p different elements."
   (b) In the language of rings $\{+, \cdot, 0, 1\}$: "the characteristic is p."
   (c) In the language of graphs $\{R\}$: "there is a path of length exactly 3 (i.e. a path with exactly 3 edges) between x and y."
   (d) In the language of orderings $\{<\}$: "every element has an immediate predecessor."

Part One

Do four of the following eight problems.

1. Using Tarski-Seidenberg—that is to say, quantifier elimination for the real ordered field in the signature $\langle 0, 1; +, -, \cdot; < \rangle$—show that the field of real numbers viewed as a pure field—that is to say, in the signature $\langle 0, 1; +, -, \cdot \rangle$—is model-complete.

2. Let L be a language and T an L-theory. Let σ_n and τ_n, for $n \in \mathbb{N}$, be sentences in this language. Show that if the infinitary conjunction $\bigwedge_{n \in \mathbb{N}} \sigma_n$ is equivalent modulo T to the infinitary disjunction $\bigvee_{n \in \mathbb{N}} \tau_n$, then these two infinitary sentences are actually equivalent modulo T to a (first-order) sentence, as follows. For each n, let T_n be the theory consisting of T, all σ_i for $i \in \mathbb{N}$, and $\neg\tau_1, \ldots, \neg\tau_n$. Show that some T_n must be inconsistent, by using an ultraproduct construction or appealing to the Compactness/Finiteness Theorem. Now deduce that the infinitary sentence $\bigvee_{n \in \mathbb{N}} \tau_n$ is first-order modulo T.

3. Prove that if $(\mathcal{M}_i | i \in \mathbb{N})$ is an elementary chain of L-structures, in the sense that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ for each i, and if $\mathcal{M}$ is the union of this chain, then $\mathcal{M}_0 \preceq \mathcal{M}$.

4. Let $\mathbf{S}_L$ the Stone space of L consisting of all complete L-theories T. Show that a theory T is finitely axiomatizable if and only if it is an isolated point of $\mathbf{S}_L$. (Recall that a point x in a topological space S is called *isolated* if the singleton $\{x\}$ is open.)

5. Give two non-isomorphic algebraically closed fields that have some nontrivial ultrapowers which are isomorphic. Substantiate all your claims about these fields.

6. Describe the maximal ideals in an infinite Cartesian product of fields in terms of ultrafilters on the index set.

7. Let S and T be L-theories such that $S_\forall \subseteq T \subseteq S$. Show that if T admits elimination of quantifiers, then $\mathrm{Mod}(T) = \mathrm{Mod}(S)$.

8. Let $\mathbb{R}_\natural$ be a nontrivial ultrapower of the reals $\mathbb{R}$ and view $\mathbb{R}$ as a subfield via the diagonal embedding. Call a non-zero element $\alpha \in \mathbb{R}_\natural$ an *infinitesimal*, if $0 < |\alpha| < r$, for every positive $r \in \mathbb{R}$; call r *infinite* if $1/\alpha$ is infinitesimal; and call α *finite* if it is not infinite (or zero). Give an example of an infinitesimal. Show that the finite elements form a ring R and the infinitesimals $\mathfrak{M}$ are an ideal in R. Show that for any element $\alpha \in \mathbb{R}_\natural$, either α is infinite or else there is a unique real number $r \in \mathbb{R}$, called the *standard part of* α, such that $\alpha - r$ is infinitesimal. Conclude that $R/\mathfrak{M} \cong \mathbb{R}$ and that $\mathfrak{M}$ is a maximal ideal.

Part Two

Do exactly four of the following eight problems.

1. Show that there is a recursive theory T extending ZFC (or PA) that proves its own inconsistency, i.e., such that $T \vdash \neg\mathrm{con}(\ulcorner T \urcorner)$.

2. Suppose N is a nondeterministic Turing machine that has no infinite run on any input. Show that for any natural number n, the lengths of possible runs of N on any input word of length n are bounded. (The function mapping n to the least such bound is called the runtime function of T.)

3. Let M be a model of a language with a binary relation symbol $\dot{R}$. Let $R = \dot{R}^M$. Suppose that $\langle |M|, R\rangle$ is a linear order, and assume that M satisfies the foundation scheme for $\dot{R}$. I.e., for every formula $\varphi(x, \vec{y})$,

$$M \models \forall \vec{y}(\exists x\ \varphi(x, \vec{y}) \quad \longrightarrow \quad \exists x \quad (\varphi(x, \vec{y}) \wedge \forall z(R(z, x) \longrightarrow \neg\varphi(z, \vec{y})))).$$

   Let D be the set of elements a of M such that $\{a\}$ is definable in M without parameters. I.e., $a \in D$ iff for some formula $\varphi(x)$, a is the unique element of M such that $M \models \varphi[a]$. Show:

   (a) D is nonempty.

   (b) The reduct of M to D is an elementary substructure of M.

   (c) If M is finite, then $D = |M|$.

   (d) D is at most countable.

4. An $\aleph_1$-tree T is $\mathbb{R}$-embeddable if there is an order-preserving function $f : T \longrightarrow \mathbb{R}$, meaning that for all $x, y \in |T|$, if $x <_T y$, then $f(x) <_{\mathbb{R}} f(y)$. Similarly, T is $\mathbb{Q}$-embeddable if there is an order-preserving function from T to $\mathbb{Q}$.

   (a) Show that if T is $\mathbb{R}$-embeddable, then T has no uncountable branch.

   (b) Show that if T is Souslin, T is not $\mathbb{Q}$-embeddable.

5. Show that any infinite model M (of a countable language) has a subset of its domain that's not definable in M, using parameters.

6. Prove that the deductive closure of a computably enumerable set of sentences, if complete, is decidable.

7. Use the Recursion Theorem to show that the set $\{e \mid \forall e' < e \quad \varphi_e \neq \varphi_{e'}\}$ is not recursively enumerable.

8. Say that an obsolete state of a Turing Machine is a state that it never enters during any computation on any input. Show that the problem of determining if a Turing Machine has an obsolete state is undecidable.