Department of Mathematics
The CUNY Graduate Center
Real Analysis Qualifying Exam
Fall 2014

Your name: ______________________________

A1	A2	A3	B1	B2	B3	B4	B5	B6	B7

Instructions

- The exam has two parts. Answer two questions from Part A and five questions from Part B. Only two+five questions will contribute to your score.
- In the above table, check the questions from each part that you would like to be graded.
- *Use only one side of each sheet.* Attach extra sheets if necessary.
- You have 3 hours to complete your work.

Conventions

- The terms "measurable," "measure," "integrable," and "almost everywhere (a.e.)" in $\mathbb{R}^n$ always refer to the σ-algebra of Lebesgue measurable sets and Lebesgue measure m.
- All measures are positive.
- If (X, μ) is a measure space, we denote by $L^p(X, \mu)$ the space of measurable functions $f : X \to \mathbb{C}$ for which

$$\|f\|_p = \begin{cases} \left(\int_X |f|^p \, d\mu \right)^{1/p} & 1 \leq p < \infty \\ \text{ess sup } |f| & p = \infty \end{cases}$$

is finite. When $X \subset \mathbb{R}^n$ and $\mu = m$, we simplify this notation to $L^p(X)$.

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PART A. THEOREMS/DEFINITIONS Answer any two of the following three questions.

A1. What does it mean for a Borel measure μ on $\mathbb{R}^n$ to be ***regular***? Give an example of a translation-invariant regular Borel measure μ on $\mathbb{R}^n$ which is not a multiple of Lebesgue measure m. Name an additional property that would ensure such a μ is a multiple of m.

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A2. State the ***open mapping theorem*** for Banach spaces. Briefly explain what this theorem implies about a bijective continuous linear map between two Banach spaces.

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A3. Define what it means for a function $[a, b] \to \mathbb{R}$ to be ***absolutely continuous***. Give an example of a continuous function which is not absolutely continuous. What can be said about the image of a set of measure zero under an absolutely continuous function?

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PART B. PROBLEMS Solve any five of the following seven problems.

B1. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and $f(0) = 0$. Prove that the sequence of powers $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if $\max_{x\in[0,1]} |f(x)| < 1$.

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B2. Suppose $\{A_k\}$ is a sequence of disjoint measurable subsets of $[0, 1]$ such that $\bigcup_{k=1}^{\infty} A_k = [0, 1]$, and $\{B_n\}$ is a sequence of measurable subsets of $[0, 1]$ such that $\lim_{n\to\infty} m(B_n \cap A_k) = 0$ for each k. Show that $\lim_{n\to\infty} m(B_n) = 0$.

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B3. Let (X, μ) be a measure space with $\mu(X) < \infty$ and suppose $f \in L^r(X, \mu)$ for some $r > 0$. Show that $f \in L^p(X, \mu)$ whenever $0 < p < r$, and

$$\lim_{p \to 0} \|f\|_p^p = \mu(\{x \in X : f(x) \neq 0\}).$$

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B4. Let (X, μ) be a measure space and $f_n, f : X \to \mathbb{C}$ be measurable. Suppose for every $\varepsilon > 0$ there is a measurable set $E \subset X$ with $\mu(E) < \varepsilon$ such that $f_n \to f$ uniformly on $X \setminus E$. Prove that $f_n(x) \to f(x)$ for μ-almost every $x \in X$.

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B5. Suppose $f \in L^p[0, \infty)$ for some $1 \leq p \leq \infty$. Show that

$$\lim_{n \to \infty} \int_0^\infty f(x) e^{-nx} \, dx = 0.$$

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B6. Verify that the function

$$f(x, y) = y\,(\sin x)\,e^{-xy}$$

is integrable on $E = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < y < 1\}$ and find $\int_E f$.

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B7. Let f be a non-zero bounded linear functional on a normed space X. Find the distance between the origin 0 and the hyperplane $\{x \in X : f(x) = 1\}$.