# Department of Mathematics
# The CUNY Graduate Center
# Real Analysis Qualifying Exam
## Spring 2014

Your name: ______________________________

| A1 | A2 | A3 | B1 | B2 | B3 | B4 | B5 | B6 | B7 |
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Instructions

- The exam has two parts. Answer two questions from Part A and five questions from Part B. Only two+five questions will contribute to your score.
- In the above table, check the questions from each part that you would like to be graded.
- *Use only one side of each sheet.* Attach extra sheets if necessary.
- You have 3 hours to complete your work.

Conventions

- The terms "measurable," "measure," "integrable," and "almost everywhere (a.e.)" in $\mathbb{R}^n$ always refer to the $\sigma$-algebra of Lebesgue measurable sets and Lebesgue measure $m$.
- All measures are positive.
- If $(X, \mu)$ is a measure space, we denote by $L^p(X, \mu)$ the space of measurable functions $f : X \to \mathbb{C}$ for which

$$\|f\|_p = \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{1/p} & 1 \leq p < \infty \\ \text{ess sup } |f| & p = \infty \end{cases}$$

is finite. When $X \subset \mathbb{R}^n$ and $\mu = m$, we simplify this notation to $L^p(X)$.

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PART A. THEOREMS/DEFINITIONS Answer any two of the following three questions.

**A1.** Define a ***separable*** metric space. For what values of $1 \leq p \leq \infty$ is the space $L^p(\mathbb{R})$ separable? Briefly explain why.

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**A2.** Carefully state ***Fubini's theorem*** on integration on the product of two measure spaces. What does this theorem imply about the 2-dimensional Lebesgue measure of the graph of a measurable function $f : \mathbb{R} \to \mathbb{R}$?

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**A3.** Assuming the Axiom of Choice, explain how one can prove that $\mathbb{R}$ (in fact every subset of $\mathbb{R}$ with positive measure) has Lebesgue non-measurable subsets.

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PART B. PROBLEMS Solve any five of the following seven problems.

**B1.** Suppose $f : [0, 1] \to \mathbb{R}$ is continuous. Show that

$$\lim_{n\to\infty} \int_0^1 nx^n f(x)\,dx = f(1).$$

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**B2.** Suppose $E \subset \mathbb{R}$ is measurable with $m(E) > 1$. Prove that there exist $x, y \in E$ such that $x - y$ is an integer.

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**B3.** Prove that there is no sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

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**B4.** Let $(X, \mu)$ be a measure space, $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$, and $h \in L^r(X, \mu)$, where $1 < p, q, r < \infty$ and $p^{-1} + q^{-1} + r^{-1} = 1$. Show that $fgh \in L^1(X, \mu)$ and

$$\|fgh\|_1 \leq \|f\|_p \, \|g\|_q \, \|h\|_r.$$

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**B5.** Suppose $(X, \mu)$ is a measure space and $\{f_n\}$ is a sequence in $L^p(X, \mu)$ for some $1 \le p < \infty$.

(i) Prove that if $\sum_{n=1}^{\infty} \|f_n\|_p$ converges, then $f_n \to 0$ a.e. in $X$.

(ii) Show by an example that the assumption in (i) cannot be relaxed to $\|f_n\|_p \to 0$.

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**B6.** Recall that $f : [a, b] \to \mathbb{R}$ is a ***Lipschitz*** function if there is a constant $C > 0$ such that $|f(x) - f(y)| \le C|x - y|$ for all $x, y \in [a, b]$. Show that the following conditions are equivalent:

(i) $f$ is Lipschitz on $[a, b]$.

(ii) $f$ is absolutely continuous on $[a, b]$ and $f' \in L^\infty[a, b]$.

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**B7.** Suppose $X, Y$ are Banach spaces and $T : X \to Y$ is linear and surjective. If there is a constant $C > 0$ such that $\|T(x)\| \geq C$ for every unit vector $x \in X$, show that $T$ is bounded.