Real Analysis
Fall 1997

Remark: Unless it is stated otherwise, measurability and the measure m or dx will always refer to Lebesgue measure on the real line $\mathbb{R}$ or some interval in $\mathbb{R}$.
Do any 7 problems.

1. a) Define what is meant by an extreme point in a convex set.
   b) Let C_0 be the space of sequences of real numbers, $a = \{a_n\}_1^\infty$, such that $\lim_{n\to\infty} a_n = 0$, with the norm $\|a\| = \sup_n |a_n|$. Show that the closed unit ball in C_0 has no extreme points.

2. Let D be a dense subset of a complete metric space (X, d). Let $f : D \to \mathbb{R}$ be a uniformly continuous function. Show that there exists a unique continuous function $F : X \to \mathbb{R}$ extending f, i.e. $F|D = f$.

3. Let f be a non-decreasing function on the real line. We know that its derivative f' exists a.e., is measurable and nonnegative. Show that on any interval $[a, b]$,
$$\int_a^b f'(x)\,dx \leq f(b^+) - f(a^+).$$
(Hint: Describe f' as a limit and use Fatou's lemma).

4. Let $f \in L^p(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $p \in [1, \infty)$. Show that $f + g \in L^p(\mathbb{R})$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

5. Let $\varphi \in (L^1(I))^*$ where $I = [0, 1]$ is the unit interval. Show that there exists $g \in L^\infty(I)$ such that
$$\phi(f) = \int_0^1 fg\,dx.$$

6. Show that there exists a transcendental number.

7. Let $A = C^0([0, 2\pi])$ be the Banach space of continouus real valued functions f on the interval $[0, 2\pi]$, satisfying $f(0) = f(2\pi)$, with the "sup" norm (i.e. $\|f\| = \sup_{x\in[0,2\pi]} |f(x)|$). Let B be the vector subspace generated by the trigonometric functions $\{\cos nx, \sin nx\}$, $x \in [0, 2\pi]$, $0 \leq n \leq \infty$. Show that B is dense in A.

8. Let f and g be the elements of $L^1(\mathbb{R})$. Show that
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\,dy$$
is in $L^1(\mathbb{R})$ and $\|f * g\|_1 =\leq \|f\|_1\|g\|_1$.

9. Let M be the space of real valued measurable functions on the unit interval $I = [0, 1]$.
   a) Define what is meant by convergence in measure in M.
   b) If $f \in M$, define
$$\sigma(f) = \int_0^1 \frac{|f(x)|}{1 + |f(x)|}\,dx,$$
   and turn M into a metric space by letting $d(f, g) = \sigma(f - g)$. Show that f_n converges to f in measure if and only if f_n converges to f in the metric d.

10. Let $f(x,t) : [0,1] \times [0,1] \to \mathbb{R}$ be a measurable function such that for every x, $t \mapsto f(x,t)$ is continuous, and furthermore there exists a function $g \in L^1([0.1])$ such that $|f(x,t)| \leq g(x)$. Show that

$$h(t) = \int_0^1 f(x,t)\,dx$$

is continuous.