Real Analysis
Fall 1998

Do any 6 problems.

1. Let $\mathbb{H}$ be a real Hilbert space, and let $\beta : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}$ be a continuous bilinear function (i.e. $|\beta(x,y)| \leq C\|x\|\,\|y\|$, $C > 0$, a real number). Show that there exists a unique bounded linear operator $T : \mathbb{H} \longrightarrow \mathbb{H}$, such that
$$\beta(x,y) = \langle Tx, y\rangle, \qquad x, y \in \mathbb{H}.$$

2. Let f be a continuous real valued function in the unit interval $[0,1]$. Given a real number $\epsilon > 0$, show that there exists rational numbers $r_0, \ldots, r_p$ such that
$$|f(x) - \sum_{i=0}^{p} r_i x^{5i}| < \epsilon$$
for every $x \in [0,1]$.

3. Let p and q be real numbers such that $1 < p < q < \infty$, and $1/p + 1/q = 1$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers such that for each sequence of complex numbers $(b_n)_{n=0}^{\infty}$ with $\sum |b_n|^q < \infty$, we know that $\sum a_n b_n$ converges. Show that $\sum |a_n|^p < \infty$.

4. a) Let δ_x be the Dirac measure concentrated at $x \in \mathbb{R}$. Let
$$\mu_n = \frac{1}{N+1} \sum_{i=0}^{N} \delta_{i/N}.$$
Compute the total variation $\|m - \mu_N\|$ where m is the Lebesgue measure.
b) If f is a continuous function on $[0,1]$, show that
$$\lim_{N\to\infty} \int_0^1 f\, d\mu_N = \int_0^1 f(x)\, dm.$$

5. Let f and g be elements of $L^1((0,\infty), \frac{dx}{x})$, i.e. $\int_0^\infty |f(x)| \frac{dx}{x} < \infty$. Define
$$f * g(x) = \int_0^\infty f(\frac{x}{t}) g(t)\, dt.$$
Show that $f * g \in L^1((0,\infty, \frac{dx}{x})$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure triple. Let $T : X \longrightarrow Y$ be a function. Let $\tilde{\mathcal{M}} = \{E \subset Y \mid T^{-1}(E) \in \mathcal{M}\}$ and define $\nu(E) = \mu(T^{-1}(E))$.
a) Show that $\tilde{\mathcal{M}}$ is a σ-algebra and that ν is a measure on $\tilde{\mathcal{M}}$.
b) If $f \in L^1(d\nu)$, show that
$$\int_Y f\, d\nu = \int_X f \circ T\, d\nu.$$

7. Let $h : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be a continuous function. For each continuous real valued function f on $[0,1]$ define

$$Tf(x) = \int_0^1 h(x,y) f(y)\, dy.$$

Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real valued functions such that $|f_n(x)| \leq M < \infty$ for all $x \in [0,1]$ and all n. Show that there exists a subsequence f_{n_k} of the f_n's such that Tf_{n_k} converges uniformly to some function on $[0,1]$.

8. Let

$$f_n(x) = \frac{1}{\sqrt{4\pi n}}\, e^{-\frac{x^2}{4n}} \qquad x \in \mathbb{R}.$$

Show that there does not exist an L^1 function g on $\mathbb{R}$ such that $f_n(x) \leq g(x)$ almost everywhere.

9. Show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hint: Use the Fourier series for $f(x) = x$ on $[0,1]$.