# REAL VARIABLES QUALIFYING EXAMINATION

**INSTRUCTIONS: Work any 8 problems. Time: three hours.**

**May 21, 1999**

1. Prove the Riemann-Lebesgue Theorem: if $f \in L^1(-\infty, \infty)$ then $\hat{f}(y) \to 0$ as $y \to \infty$.
2. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and satisfies $\int_0^1 f(x)x^n\,dx = 0$ for all integers $n \geq 0$. Prove that $f(x) \equiv 0$. (Hint. Use the Weierstrass Approximation Theorem).
3. When $(X, \mu)$ is a measure space satisfying $\mu(X) < \infty$, show that $L^2(X) \subset L^1(X)$. Discuss the case in which $\mu(X) = \infty$.
4. Suppose $\phi_1, \phi_2, \ldots$ is an orthonormal sequence in a Hilbert space $H$, and $f \in H$. Show that for any $n$, the norm of $f - \sum_{j=1}^{n} c_j\phi_j$ is minimized by choosing $c_j = (f, \phi_j)$.
5. Discuss in outline, with pertinent definitions but without proofs, the phenomena of recurrence and transience for random walks on $\mathbb{Z}^2$ and $\mathbb{Z}^3$.
6. Suppose $S_1, S_2, \ldots$ are measurable subsets of $\mathbb{R}^1$ and that the sum of their measures is finite. Let $A$ be the set of points in infinitely many of the $S_n$'s. Show that $A$ is of measure zero.
7. Suppose $X$ is a complete metric space, and that $A_1, A_2, \ldots$ is a sequence of open dense subsets of $X$. Show that $\bigcap A_n$ is a dense subset of $X$. (the Baire Category Theorem).
8. In the above problem, suppose $X = [0,1]$. Discuss in detail the cardinality of $\bigcap A_n$.

9. Show that any continuous linear functional $T$ on a Hilbert space $H$ is of the form $T(f) = (f, g)$, for some $g \in H$.

10. Assuming Beppo Levi's theorem (the monotone convergence theorem), prove Fatou's theorem: if $\{f_n\}$ is a sequence of non-negative functions in $L^1$ for which $\int f_n \leq M$ and $f_n \to f$ a.e., then $f \in L^1$ and $\int f \leq \liminf \int f_n$.

11. Show by example that strict inequality is possible in Fatou's theorem.

12. Outline a proof that $L^1$ is complete.

13. State (do not prove) Weyl's criterion for equidistribution (mod 1) of a sequence. Use the Weyl criterion to show that if $\alpha$ is irrational, then the sequence $\alpha, 2\alpha, 3\alpha, \ldots$ is equidistributed mod 1.

14. Outline a proof of the Central Limit Theorem.

15. Show that the unit square is a continuous image of the Cantor set.