REAL VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work any 8 problems. Time: three hours.

May 21, 1999

1. Prove the Riemann-Lebesgue Theorem: if \( f \in L^1(-\infty, \infty) \) then \( \hat{f}(y) \to 0 \) as \( y \to \infty \).

2. Suppose \( f : [0, 1] \to \mathbb{R} \) is continuous and satisfies \( \int_0^1 f(x)x^n \, dx = 0 \) for all integers \( n \geq 0 \). Prove that \( f(x) \equiv 0 \). (Hint. Use the Weierstrass Approximation Theorem).

3. When \( (X, \mu) \) is a measure space satisfying \( \mu(X) < \infty \), show that \( L^2(X) \subset L^1(X) \). Discuss the case in which \( \mu(X) = \infty \).

4. Suppose \( \phi_1, \phi_2, \ldots \) is an orthonormal sequence in a Hilbert space \( H \), and \( f \in H \). Show that for any \( n \), the norm of \( f - \sum_{j=1}^n c_j \phi_j \) is minimized by choosing \( c_j = (f, \phi_j) \).

5. Discuss in outline, with pertinent definitions but without proofs, the phenomena of recurrence and transience for random walks on \( \mathbb{Z}^2 \) and \( \mathbb{Z}^3 \).

6. Suppose \( S_1, S_2, \ldots \) are measurable subsets of \( \mathbb{R}^4 \) and that the sum of their measures is finite. Let \( A \) be the set of points in infinitely many of the \( S_n \)'s. Show that \( A \) is of measure zero.

7. Suppose \( X \) is a complete metric space, and that \( A_1, A_2, \ldots \) is a sequence of open dense subsets of \( X \). Show that \( \bigcap A_n \) is a dense subset of \( X \). (the Baire Category Theorem).

8. In the above problem, suppose \( X = [0, 1] \). Discuss in detail the cardinality of \( \bigcap A_n \).
9. Show that any continuous linear functional $T$ on a Hilbert space $H$ is of the form $T(f) = (f, g)$, for some $g \in H$.

10. Assuming Beppo Levi's theorem (the monotone convergence theorem), prove Fatou's theorem: if $\{f_n\}$ is a sequence of non-negative functions in $L^1$ for which $\int f_n \leq M$ and $f_n \to f$ a.e., then $f \in L^1$ and $\int f \leq \lim \inf \int f_n$.

11. Show by example that strict inequality is possible in Fatou's theorem.

12. Outline a proof that $L^1$ is complete.

13. State (do not prove) Weyl's criterion for equidistribution (mod 1) of a sequence. Use the Weyl criterion to show that if $\alpha$ is irrational, then the sequence $\alpha, 2\alpha, 3\alpha, \ldots$ is equidistributed mod 1.


15. Show that the unit square is a continuous image of the Cantor set.