**Real Analysis Qualifying Exam** **Fall 2008**

Do any eight problems. Only the first eight problems in your exam booklet will be graded. Clearly cross out any attempts at solving a problem you do not wish to include amongst the eight to be graded.

1. Show that for all integers $n$ and any positive numbers $x_1, \dots, x_n$
$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1 + x_2 \cdots + x_n}{n}.$$

2. Let $X = (X, \mathcal{M})$ be a measurable space and let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. Prove that the functions $\sup_n f_n : X \to (-\infty, \infty]$ and $\limsup_n f_n : X \to (-\infty, \infty]$ are also measurable.

3. By an *interval* in $\mathbb{R}$ one means a subset of any one of the forms $[a, b]$, $(a, b]$, $[a, b)$ or $[a, b]$, where $a \le b$ (and $(a, a) := (a, a] := [a, a) := \emptyset$). The real numbers $a$ and $b$ are the "endpoints" of all of these intervals.

   By an *interval* in $\mathbb{R}^n$ when $n > 1$ one means a Cartesian product
$$I := I_1 \times I_2 \times \cdots \times I_n \tag{i}$$
   in which each $I_j$ is an interval in $\mathbb{R}$. Prove that the outer measure of $I$ is given by $\prod_{j=1}^{n}(b_j - a_j)$, where $a_j$ and $b_j$ denote the endpoints of $I_j$.

4. Let $H$ be a separable Hilbert space with inner product $(\cdot, \cdot)$ and $\{e_n\}_{n=1}^{\infty}$ an orthonormal set in $H$, and let $x, y$ in $H$ be given by $x = \sum_{n=1}^{\infty} x_n e_n$ and $y = \sum_{n=1}^{\infty} y_n e_n$. Show that $(x, y) = \sum_{n=1}^{\infty} x_n y_n$.

5. Let $\nu$ and $\mu$ be two measures on the same measurable space $(X, \mathbf{S})$. Explain what it means to say that $\nu$ is absolutely contiinuous with respect to $\mu$ and what it means to say that $\nu$ is singular with respect to $\mu$. State both the Lebesgue Decomposition Theorem and the Radon Nykodym Theorem.

6. Let $\{a_n\}_{n \ge 1}$ and $\{b_n\}_{n \ge 1}$ be sequences in $[-\infty, \infty]$ satisfying $a_n \le b_n$ for all $n \ge 1$. Prove that $\liminf_{n \to \infty} a_n \le \liminf_{n \to \infty} b_n$.

7. State and prove the Hahn–Banach Theorem.

8. Let $X = (X, \mathcal{M}, \mu)$ be a measure space with positive measure $\mu$ and let $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

   Now suppose $X = [a, b] \subset \mathbb{R}$ and $\mathcal{M}$ and $\mu$ are induced by Lebesgue measure. Prove that $F : x \in [a, b] \mapsto \int_a^x f(t) \, d\mu$ is continuous and of bounded variation.

9. Let $(X, \|\cdot\|)$ be a normed linear space over the complex numbers. What do we mean by the dual space of $(X, \|\cdot\|)$? Show that it is a Banach space.

10. Let $X = (X, \mathcal{M}, \mu)$ be a measure space with positive measure $\mu$, and let $p$ and $q$ be conjugate exponents with $1 < p < \infty$. (In other words, $1/p + 1/q = 1$.) Prove *Hölder's Inequality*, i.e., prove that when $f, g : X \longrightarrow [0, \infty]$ are measurable one has

$$\int_X fg \, d\mu \leq \left( \int_X f^p \, d\mu \right)^{1/p} \left( \int_X g^q \, d\mu \right)^{1/q}.$$

11. A collection $\mathcal{M}$ of subsets of a non-empty set $X$ is called a *monotone class* if $\cup_n A_n \in \mathcal{M}$ and $\cap_n B_n \in \mathcal{M}$ whenever $A_n, B_n \in \mathcal{M}$ and $A_n \subset A_{n+1}$ and $B_n \supset B_{n+1}$, for $n = 1, \ldots$.

    Give an example of a monotone class which contains $\emptyset$ and $X$, but is not a $\sigma$-algebra. Prove your assertions.

12. Let $X$ be a locally compact space. A subset $E \subset X$ is *$\sigma$-bounded* if there is a sequence $\{C_n\}_{n=1}^{\infty}$ of compact sets such that $E \subset \cup_{n=1}^{\infty} C_n$. Prove that the collection of all $\sigma$-bounded sets is a $\sigma$-ring, i.e., that it is closed under set differences and countable unions.

13. State the Monotone Convergence Theorem and the Dominated Convergence Theorem. Use the Monotone Convergence Theorem to prove the Dominated Convergence Theorem.

14. Show that if a real valued function $f$ on a Hilbert space $H$ is linear and continuous then there exists a unique $h \in H$, such that $f(x) = (x, h)$ for all $x \in H$. ( $( \cdot , \cdot )$ is an inner product on $H$.)

15. Find the surface area of the unit sphere in $R^k$, $2 \leq k < \infty$.

16. Prove *Egoroff's Theorem*: Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of measurable functions which converges to a function $f : \mathbb{R} \to \mathbb{R}$ at each point of a set $E \subset \mathbb{R}$ of finite measure. Then for each $\epsilon > 0$ there is a subset $F \subset E$ such that $m(F) < \epsilon$ and $f_n$ converges to $f$ uniformly on $E \setminus F$.