REAL VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work any eight problems. Time: three hours.

September 2, 2009

1. Show that a function which is continuous on a compact metric space \( X \) is uniformly continuous on \( X \). Give an example to show that compactness is essential.

2. Suppose \( S_1, S_2, \ldots \) are measurable subsets of \( \mathbb{R}^1 \) and that the sum of their measures is finite. Let \( A \) be the set of points in infinitely many of the \( S_n \)’s. Show that \( A \) is of measure zero.

3. Suppose \( \psi_1, \psi_2, \ldots \) are linearly independent elements of a Hilbert space \( H \). Prove that the \( \psi_n \)’s can be orthonormalized. I.e., there exists an orthonormal sequence \( \varphi_1, \varphi_2, \ldots \) of elements of \( H \), such that for each \( N \), \( \text{span}(\varphi_1, \ldots, \varphi_N) = \text{span}(\psi_1, \ldots, \psi_N) \). Explicitly compute \( \varphi_1, \varphi_2, \) and \( \varphi_3 \) when \( H = L^2[-1, 1] \) and \( \psi_1, \psi_2, \psi_3, \psi_4, \ldots = 1, x, x^2, x^3, \ldots \).

4. Let \( \|f\| = \sup_{x \in [0, 1]} |f(x)| \) be the sup norm on \( C[0, 1] \), the space of real-valued continuous functions on \( C[0, 1] \). With this norm, \( C[0, 1] \) is a Banach space. Prove or disprove: This norm arises from an inner product on \( C[0, 1] \), i.e., \( C[0, 1] \) is also a Hilbert space with this norm.

5. State and prove Ascoli’s theorem for families of real-valued functions on a compact metric space \( X \). For \( X = [0, 1] \), give an example of an equicontinuous family, and of a family which is not equicontinuous.

6. Assuming Beppo Levi’s theorem (the Monotone Convergence Theorem), prove Fatou’s theorem: if \( \{f_n\} \) is a sequence of non-negative functions in \( L^1 \) for which \( \int f_n \leq M \) and \( f_n \to f \) a.e., then \( f \in L^1 \) and \( \int f \leq \lim \inf \int f_n \).

7. Prove the following strengthened version of the Baire Category Theorem for the space \([0, 1] \): A countable intersection of dense open subsets of \([0, 1] \) has cardinality \( \aleph \).
8. Let \( \|x\| \) denote the distance of \( x \) from the nearest integer. Suppose \( \sum_{n=1}^{\infty} a_n \) is an absolutely convergent series, and \( 0 < \alpha < 1 \). Show that the series defining \( f(x) = \sum_{n=1}^{\infty} a_n \|nx\|^{-\alpha} \) converges for almost all \( x \in \mathbb{R}^1 \). Hint: By periodicity, it is enough to consider \( x \in [0, 1] \). Try to use the Monotone Convergence Theorem.

9. Prove the Riemann-Lebesgue Lemma: If \( f(x) \in L^1(-\infty, \infty) \), and \( \hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy} \, dx \), then \( \hat{f}(y) \to 0 \) as \( y \to \infty \). Can you give an example for which \( \hat{f}(y) \) is not in \( L^1(-\infty, \infty) \)?

10. Define what it means for a function to be of bounded variation on \([a, b]\), and what it mean for a function to be absolutely continuous on \([a, b]\). Discuss, with examples if possible, the relationship between these two classes. Specifically, must a function of bounded variation be absolutely continuous? Must an absolutely continuous function be of bounded variation?