REAL VARIABLES QUALIFYING EXAMINATION

INSTRUCTIONS: Work any 8 problems. Time: three hours.

May 22, 2006

1. Show how to construct a non-Lebesgue measurable subset of the unit circle and supply proofs of your assertions.

2. A sequence $\{a_1, a_2, \ldots\}$ is said to be Cesáro convergent if the associated sequence of averages $S_n = (1/n)(a_1 + \cdots + a_n)$ is convergent. Show that $a_n \to L \Rightarrow S_n \to L$, and give an example to show that the reverse is not true.

3. Suppose $E_1 \subset E_2 \subset \ldots$ is a nested sequence of measurable subsets of $[a, b]$. Prove that $m(\bigcup_n E_n) = \lim_{n\to\infty} m(E_n)$.

4. Define absolute continuity of a measure with respect to another, and state the Radon-Nikodym theorem.

5. Prove that the Fourier series of $f \in L^2[0, 1]$ converges to f in the L^2 norm.

6. Suppose $1 \leq p \leq \infty$. Prove that $L^p[a, b]$ is complete.

7. Suppose $\{a_1, a_2, \ldots\}$ is a sequence of real numbers for which $\sum_{n=1}^{\infty} |a_n| < \infty$. Show that $f(x) = \sum_{n=1}^{\infty} a_n \cos 2\pi nx$ is continuous and periodic on $[0, 1]$, and that $f(x)$ has at least one zero on $[0, 1]$. (Hint for 2nd part: consider the implications of the fact that $a_0 = 0$).

8. Suppose $f \in L^1[a, b]$. Show that for every $\epsilon > 0$, there is a $\delta > 0$ such that if E is any measurable subset of $[a, b]$ with $m(E) < \delta$, then $\int_E f < \epsilon$.

9. Suppose $A \subset R^1$ is Lebesgue measurable of measure $m(A)$, and suppose $0 < \beta < m(A)$. Show that there is a Lebesgue measurable subset $B \subset A$ such that $m(B) = \beta$. (Hint: show that without loss of generality one can assume $A \subset [-n, n]$ for sufficiently large n; then consider the function $f : x \to m([-n, x] \cap A)$).

10. True or false: any open dense subset E of $[0, 1]$ is of full measure, i.e., $m(E) = 1$. Either prove this or give a counterexample.

11. Suppose X is a compact metric space, and $\mathcal{O}$ an open cover, i.e., $X \subset \bigcup_{U \in \mathcal{O}} U$, where each U is an open subset of X. Show that there is a Lebesgue number associated with the cover, i.e., an $\epsilon > 0$ such that any open ball of radius $\leq \epsilon$ in X is contained in at least one of the U's.

12. State and prove Ascoli's Theorem for a compact metric space.

13. Suppose $f \in L^1[0, 1]$ and $\int_0^1 x^{2n} f(x)\, dx = 0$ for $n = 0, 1, 2, \ldots$. Prove that $f = 0$ a.e.

14. Suppose $f_n \downarrow 0$ is a sequence of continuous functions on $[a, b]$ which monotonically decreases to 0. Show that $f_n \downarrow 0$ uniformly.

15. Suppose $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$ is the Fourier series of an L^1 function f. Show that the Nth partial sum, $S_N(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$, is equal to

$$\int_0^1 f(t) D_N(t - x)\, dt,$$

where $D_N(u)$ is the Nth Dirichlet kernel, i.e.,

$$D_N(u) = \frac{\sin 2\pi \left(N + \frac{1}{2}\right) u}{\sin \pi u}.$$

16. True or False: the set of rationals in R^1 is the intersection of countably many open sets. Give a detailed proof for your answer. It is not enough to simply quote a theorem.