TOPOLOGY QUALIFYING EXAM - SEPTEMBER 2000

INSTRUCTIONS: THE EXAM CONSISTS OF THREE PARTS, EACH WITH FOUR QUESTIONS. ANSWER AT LEAST TWO QUESTIONS FROM EACH OF THE THREE PARTS AND EIGHT QUESTIONS OVERALL.

PART I

1) Let $X$ be a space*, $I = [0, 1]$ with the Euclidean topology, and consider $X \times I$ with the Tychonoff topology.
   (i) Show that $X$ is metrizable $\iff X \times I$ is metrizable.
   (ii) Show that $X$ has a countable basis $\iff X \times I$ has a countable basis.

2) Let $f : X \to Y$ be a map* from a space $X$ to a space $Y$.
   (i) If $f$ is one-to-one and $Y$ is Hausdorff, must $X$ be Hausdorff? Explain.
   (ii) If $f$ is onto and $X$ is Hausdorff, must $Y$ be Hausdorff? Explain.

3) Give examples, with brief explanations, of:
   (i) An open, onto map which is not a closed map.
   (ii) A closed, onto map which is not an open map.
   (iii) A quotient map which is neither an open map nor a closed map.

4) Let $X_n$ be the one-point union of $n$ copies of the circle, $S^1$, $1 \leq n \leq \infty$.
   [Thus $X_n$ is obtained from a disjoint union of $n$ copies of $S^1$ by choosing a basepoint in each $S^1$ and identifying these basepoints with each other. A subset $U$ of $X_n$ is open $\iff$ the intersection of $U$ with each $S^1$ is open in that $S^1$.
   ]
   (i) Prove that $X_n$ is compact $\iff n < \infty$.
   (ii) Prove that $X_n$ is connected for all $n$. 
PART II

5) (i) Sketch calculations of the fundamental groups of $X_2$ (in the notation of Question 4) and the torus $T^2 = S^1 \times S^1$.
(ii) $X_2$ may be viewed as a subspace of $T^2$. Is $X_2$ a retract*** of $T^2$? Explain.

6) (i) Sketch calculations of the fundamental groups of $S_2$, the surface of genus 2, and $\mathbb{R}P^2$, the real projective plane.
(ii) Is it possible for $S_2$ to be a covering space of $\mathbb{R}P^2$? Explain.

7) (i) Sketch a calculation of the fundamental group of $\mathbb{R}P^3 \times S^1$, where $\mathbb{R}P^3$ is 3-dimensional real projective space.
(ii) Prove that the spaces $\mathbb{R}P^3 \times S^1$, $S^3 \times S^1$ (where $S^n$ denotes the $n$-dimensional sphere) are not homeomorphic but that their universal covering spaces are homeomorphic.

8) (i) Describe all the connected covering spaces of $\mathbb{C}P^2 \times S^1$, where $\mathbb{C}P^2$ is the complex projective plane.
(ii) Is the universal covering space of $\mathbb{C}P^2 \times S^1$ contractible? Explain.
PART III

9) (i) Sketch a calculation of the integral homology groups of $\mathbb{R}P^2$.
    (ii) State the Universal Coefficient Theorem for homology groups.
    (iii) Using (i) and (ii), derive a calculation of the mod 2 homology groups of $\mathbb{R}P^2$.

10) Sketch a calculation of the integral homology groups of $X_n$ (in the notation of Question 4).

11) Let $A$ be a subspace of a contractible space $X$.
    (i) Prove that $H_n(A)$ is isomorphic to $H_{n+1}(X, A)$ if $n > 0$.
    (ii) What is the correct statement if $n = 0$?

12) (i) State the Kunneth Theorem for homology groups.
    (ii) Prove that the spaces $S^2 \times S^4$, $S^3 \times S^3$ are not homeomorphic.
FOOTNOTES

* Throughout, space is used as an abbreviation for topological space.

** Throughout, map is used as an abbreviation for continuous map.

*** A subspace $R$ of a space $X$ is a retract if there is a map from $X$ to $R$ which, when restricted to $R$, is the identity map from $R$ to itself.