Topology Qualifying Exam
Spring 2000

Instructions. The exam consists of three parts. Answer at least two questions from each of the three parts and eight questions overall.

PART I

1) Let $X = \Pi X_\lambda$ be a Cartesian product of non-empty spaces* with the Tychonoff topology.

(i) Show that X is Hausdorff $\Leftrightarrow$ each X_α is Hausdorff.

(ii) Show that X is path-connected $\Leftrightarrow$ each X_α is path-connected.

2) Give examples, with brief explanations, of:

(i) A connected space which is not path-connected.

(ii) A subspace W of a space X with W compact, but not closed in X.

(iii) A Hausdorff, non-metrizable space

(iv) A quotient map** $q : X \to Y$ which is not an open map.

3) Define an equivalence relation on $\mathbb{R}$, the real numbers with the Euclidean topology, by setting $x \sim y \Leftrightarrow [x] = [y]$, where

$$[u] = \text{ the greatest integer } \leq u.$$

Denote by Q the resulting quotient space and by $q : \mathbb{R} \to Q$ the resulting quotient map.

(i) Show that Q is a countably infinite, connected space.

(ii) Is Q compact? Explain.

(iii) Is Q metrizable? Explain.

4) Let ΣX, the suspension of X, be the quotient space of the Cartesian product $X \times [0, 1]$ obtained by making the identifications $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$ for all x, x' in X. View X as a subspace of ΣX by means of the map which takes x into the equivalence class of $(x, \frac{1}{2})$.

(i) Show that ΣX is compact $\Leftrightarrow$ X is compact.

(ii) Show that ΣX is Hausdorff $\Leftrightarrow$ X is Hausdorff.

* Throughout, **space** is used as an abbreviation for **topological space**.

** Throughout, **map** is used as an abbreviation for **continuous map**.

5. Let $p : X \to Y$ be a quotient space mapping. Let B be a closed subspace of Y and $A = p^{-1}(B)$. Prove or disprove: $q : A \to B$ is a quotient space mapping where $q(a) = p(a), \forall a \in A$.

6.
- Give an example of a continuous bijection which is not a homeomorphism.
- Show that a continuous bijection is a homeomorphism if the source is compact and the target is Hausdorff.

PART II

7. Let ΣX be the suspension of X (see Question 4) for the definition).
   (i) **Sketch** a proof of the following: If X is path-connected, then ΣX is simply-connected.
   (ii) Show that if X is contractible, then ΣX is also contractible.

8. (i) **Sketch** calculations of the fundamental groups of the circle, S^1, and n-dimensional real projective space, $\mathbb{R}P^n$, $n > 0$, using covering space theory (or otherwise).
   (ii) Regard $\mathbb{R}P^9$ as a subspace of 9-dimensional complex projective space, $\mathbb{C}P^9$, and hence $\mathbb{R}P^9 \times S^1$ as a subspace of $\mathbb{C}P^9 \times S^1$. Is $\mathbb{R}P^9 \times S^1$ a retract*** of $\mathbb{C}P^9 \times S^1$? Explain.

9. Let $X = S^1 \vee S^2$, the one-point union of the circle and the 2-dimensional sphere. [Thus X is the quotient space obtained from the disjoint union of the circle and the 2-dimensional sphere by identifying a basepoint in the circle with a basepoint in the 2-dimensional sphere. The basepoint in X is the equivalence class of these two identified points - call it p.] $q \neq p$
   (i) **Sketch** a calculation of the fundamental group of $X \setminus \{q\}$, X with a single point removed. [The answer may depend on where q lies!]
   (ii) Let $q_1 \in S^1$, $q_2 \in S^2$ and suppose $q_1 \neq p \neq q_2$. Is there a homeomorphism $f : X \to X$ such that $f(q_1) = f(q_2)$? Explain.

[You may assume as known the fundamental group of S^1 and use this information in your calculation.]

10. **Sketch** calculations of the fundamental groups of:
   (i) K, the Klein Bottle****.
   (ii) $K \setminus \{k\}$, the Klein Bottle with a single point removed.

[You may assume as known the fundamental group of S^1 and use this information in your calculations.]

*** A subspace R of a space X is a **retract** if there is a map from X to R which, when restricted to R, is the identity map from R to itself.

**** The **Klein Bottle** is the quotient space of the square $[0,1] \times [0,1]$ obtained by making the identifications $(s,0) \sim (s,1)$, $(0,t) \sim (1,1-t)$, $0 \leq s \leq 1$, $0 \leq t \leq 1$.

11. Let $p : X \to Y$ be a 3-sheeted covering space mapping between compact, connected, surfaces.

- Suppose the surfaces are orientable and Y is a surface of genus 5; what is the genus of X?
- Suppose Y is a non-orientable surface of Euler characteristic -1; what can be said about X?

12.
- What are all the connected covering spaces of $T = S^1 \times S^1$?
- What are all the connected covering spaces of $P_2(R)$?

PART III

13. **Sketch a calculation of the integral homology groups of the orientable surface of genus 2 (connected sum of two tori).**

14. Let J be a subspace of S^2 Suppose J is homeomorphic to a circle. What are the homology groups of $S^2 - J$? Outline the calculation of these groups.

15. **(i) Describe $H_*(\mathbb{R}P^n; A)$, the homology of n-dimensional real projective space with coefficients $A = \mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Q}$.**
(ii) Is $\mathbb{R}P^5 \vee S^5$ a retract of $\mathbb{R}P^6 \vee S^6$? Explain.

16. **Let ΣX be the suspension of X (see Question 4) for the definition).**
(i) Sketch a proof of the following: If $k > 0$, then $H_k(X; A)$ is isomorphic to $H_{k+1}(\Sigma X; A)$ for any coefficients A.
(ii) Let X be a countably infinite discrete space. Calculate $H_*(\Sigma X; \mathbb{Z})$.

17. **For a positive integer d, denote by $C(d)$ a path-connected, simply-connected CW-complex whose integral homology group $H_k(C(d); \mathbb{Z})$ is isomorphic to $\mathbb{Z}/d$ for $k = 2$, and is 0 for $k > 2$.**
(i) Calculate $H_*(C(2) \times C(5); A)$ for $A = \mathbb{Z}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/5$.
(ii) Calculate $H_*(C(3) \times C(3); A)$ for the same A as in (i).

18. Give a careful statement of and sketch the proof of the theorem suggested by the formula

$$\chi(X) = \Sigma(-1)^p \alpha_p = \Sigma(-1)^p \beta_p.$$

19. Prove that if R is a retract of X there are, $\forall p$, isomorphisms as follows.

$$H_p(R) \oplus H_p(X, R) \cong H_p(X).$$