CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM

Spring 2022
3 hours

Instructions. The exam consists of two parts. Choose a total of six problems, including two from each part. Indicate on the front cover of the exam booklet the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part 1

1. A group G acts faithfully on a set S of five elements, and there are two orbits, an orbit with 3 elements and an orbit with 2 elements. What are the possibilities for G?

2. Prove that is H has finite index n in the group G, then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

3. Determine all the ideals of the ring $\mathbf{Z}[x]/(2, x^3 + 1)$

4. Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Let B be the subgroup of A generated by $u = (0, 3, 6), v = (6, -3, 0),\ w = (0, 3, 12)$. Write A/B as a direct sum of cyclic groups.

5 Let G be a group of order 30. Prove that G has a normal subgroup isomorphic to the cyclic group of order 15.

6. Give an example of an abelian group A such that $A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$ for all primes p but $A \neq 0$ or prove there is no such abelian group.

Part 2

7. Let p be a prime number. Let F_p denote the finite field with p elements. Let $f(x)$ denote the polynomial $x^{p^4} - x$. Prove that all irreducible factors of $f(x)$ are distinct. Irreducible factors of which degrees does $f(x)$ have?

8. Determine the Galois group of the splitting field (over Q) of

$x^4 - 14x^2 + 9$, and write down the lattice of subgroups and corresponding subfields. Which subfields are Galois over Q?

9. Let G be a finite abelian group. Prove that there is a subfield K of a cyclotomic field with $Gal(K/\mathbb{Q}) \cong G$.

10. Find one representative of each conjugacy class of 5×5 matrices over Q with characteristic polynomial $(x^3 - 1)(x^2 - 1)$.

11. Suppose that every left ideal in a ring R is projective. Prove that every quotient of an injective left R-module is injective. (Hint: use the Baer criterion.)

12. Suppose I is an ideal in $F[x_1, \ldots, x_n]$ (F is a field) generated by a (possible infinite) set $\mathcal{S}$ of polynomials. Prove that a finite subset of the polynomials of $\mathcal{S}$ suffice to generate I.