Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam
May 2022

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

Notation

- $\mathbb{R}$: Set of all real numbers
- $\mathbb{C}$: The complex plane
- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: The open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: The upper half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$

PART A. ANSWER ANY TWO OF THE FOLLOWING THREE QUESTIONS.

A1. Give precise statements of the following: (i) Picard's little theorem; (ii) Picard's great theorem; (iii) Mittag-Leffler's theorem; (iv) the Weierstrass theorem on zeros of a holomorphic function in a region; (v) the Riemann mapping theorem.

A2. State and prove the Schwarz lemma.

A3. What does it mean for a point on the unit circle $\{z : |z| = 1\}$ to be a *singular point* of a holomorphic function $f : \Delta \to \mathbb{C}$? Prove that if the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence 1, then f has at least one singular point on the unit circle.

PART B. SOLVE ANY TWO OF THE FOLLOWING THREE PROBLEMS.

B1. Let $f : \Delta \to \Delta$ be holomorphic, $f(1/3) = 0$ and $f'(1/3) = 0$. Show that $|f(0)| \leq 1/9$.

B2. Let $f : \mathcal{U} \to \mathcal{U}$ be holomorphic. Prove that

$$\left|\frac{f(z) - f(i)}{f(z) - \overline{f(i)}}\right| \leq \left|\frac{z - i}{z + i}\right|$$

for all $z \in \mathcal{U}$.

B3. Suppose f and g are entire functions such that $f^2 + g^2 = 1$ everywhere in $\mathbb{C}$. Show that $f = \cos(h)$ and $g = \sin(h)$ for some entire function h.

Part C. Solve any four of the following seven problems.

C1. Show that if an entire function f maps the real axis into itself and the imaginary axis into itself, then f must be an odd function.

C2. Let f be a non-constant entire periodic function. Prove that f must have infinitely many fixed points.

C3. Let $p_1, \ldots, p_n \in \Delta$. If the polynomial $f(z) = (z - p_1) \cdots (z - p_n)$ satisfies $|f(z)| \leq 1$ for $|z| = 1$, show that $p_1 = \cdots = p_n = 0$.

C4. Suppose $\{f_n\}$ is a sequence of holomorphic functions in a region $\Omega \subset \mathbb{C}$ such that $f(z) = \lim_{n \to \infty} f_n(z)$ exists for every $z \in \Omega$. If $|f_n - 1| \geq 1$ for all n, show that $f_n \to f$ uniformly on compact subsets of Ω.

C5. Suppose $U \subset \mathbb{C}$ is a region and $u : U \to \mathbb{R}$ is a non-constant harmonic function. Show that the set $\{z \in U : \nabla u(z) = 0\}$ has no accumulation point in U (as usual, ∇u denotes the gradient of u).

C6. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected region and $z_0 \in \Omega$. The *conformal radius* $\rho = \rho(\Omega, z_0)$ is defined as $|f'(0)|$, where $f : \Delta \to \Omega$ is any Riemann map sending 0 to z_0 (it is easily seen that ρ is independent of the choice of f). Show that

$$1 \leq \frac{\rho}{\operatorname{dist}(z_0, \partial\Omega)} \leq 4,$$

where $\operatorname{dist}(z_0, \partial\Omega)$ denotes the Euclidean distance between z_0 and the boundary $\partial\Omega$.

C7. Let $\lambda > 1$. Show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the right half-plane $\{z : \operatorname{Re}(z) > 0\}$. Prove that this solution must be real. What happens to the solution as $\lambda \to 1$?