Logic Qualifying Exam

Three Parts

May 25, 2022

Part Zero (16 points)

Answer each of the following questions fully. None should take more than a few paragraphs, and some may need less than that. It is not advisable to spend more than 30 minutes on Part Zero.

1. State the Soundness and Completeness Theorems for first-order logic, and describe a proof of each. (A paragraph or two for each proof should suffice.)

2. Without using the Incompleteness Theorems or the Undefinability of Truth, explain how the undecidability of the Halting Problem implies that for every decidable axiom set A of sentences from the theory of $(\mathbb{N}, 0, 1, +, \cdot, <)$, the set $\mathrm{Cn}(A)$ of all consequences of A is incomplete.

3. State the Compactness Theorem for first-order logic (also known as the Finiteness Theorem) correctly, and use it to sketch a proof that, for linear orders (in the language with just $<$ and equality), the property of being a well-order is not first-order definable.

4. Prove in **ZFC** that the strong limit cardinals form a closed unbounded class. (A cardinal κ is a *strong limit cardinal* if every cardinal $\lambda < \kappa$ satisfies $2^\lambda < \kappa$.) Try to point out where specific axioms of **ZFC** are used in your proof.

Part One (42 points)

Do four of the following eight problems. In what follows, L always denotes a language, $\mathcal{M}$ an L-structure with underlying set M, and T an L-theory.

1. Using Tarski-Seidenberg—that is to say, quantifier elimination for the real ordered field in the language $\langle 0, 1; +, -, \cdot; < \rangle$—to show that the field of real numbers viewed as a pure field—that is to say, in the language $\langle 0, 1; +, -, \cdot \rangle$—is model-complete.

2. Let σ_n and τ_n, for $n \in \mathbb{N}$, be (countably many) sentences in L. Suppose that for any model $\mathcal{M}$ of T, one of the σ_n (where n may depend on the model $\mathcal{M}$) holds in $\mathcal{M}$ if and only if all τ_m hold in $\mathcal{M}$. Show that there is then a sentence φ, so that φ holds in a model $\mathcal{M} \models T$ if and only if all τ_m hold in $\mathcal{M}$. Is the countability assumption necessary?

3. Prove that if $(\mathcal{M}_i | i \in \mathbb{N})$ is an elementary chain of L-structures, in the sense that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ for each i, and if $\mathcal{M}$ is the union of this chain, then $\mathcal{M}_0 \preceq \mathcal{M}$.

4. Let $\mathbf{S}_L$ denote the Stone space of L, consisting of all complete L-theories T. Show that a theory T is finitely axiomatizable if and only if it is an isolated point of $\mathbf{S}_L$. (Recall that a point x in a topological space S is called *isolated* if the singleton $\{x\}$ is open.)

5. Let $\mathbb{Q}^{\text{alg}}$ and $\mathbb{Q}(t)^{\text{alg}}$ be the respective algebraic closures of the rationals $\mathbb{Q}$ and the field of Laurent series in one variable t (i.e., the field of fractions of $\mathbb{Q}[t]$). Are $\mathbb{Q}^{\text{alg}}$ and $\mathbb{Q}(t)^{\text{alg}}$ isomorphic? Are they elementarily equivalent? What about their ultrapowers with respect to some fixed ultrafilter on $\mathbb{N}$?

6. A collection $\Phi(x)$ of L-formulae in one free variable x with parameters from M is called *finitely satisfiable* in $\mathcal{M}$, if for any finite number of formulae $\varphi_1, \dots, \varphi_n \in \Phi$, there exists $a \in M$ such that $\mathcal{M} \models \varphi_i(a)$, for $i = 1, \dots, n$. Show that if $\Phi(x)$ is finitely satisfiable in $\mathcal{M}$, then there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$, such that Φ is satisfiable in $\mathcal{N}$ (meaning that there exists $b \in N$ such that $\mathcal{N} \models \varphi(b)$, for all $\varphi \in \Phi$).

7. Suppose L contains a binary relation symbol $<$. Let us call $\mathcal{M}$ an *ordered* structure, if its reduct to $\{<\}$ is a model of the theory of dense linear orders without endpoints. View any such ordered structure in its order topology. Let us say that the topological boundary $\partial D := \text{cl}(D) \setminus \text{int}(D)$ of a definable subset $D \subset M$ is *secure*, if for any two points in M, one belonging to D and one not, there is a boundary point between them. Show that if $\mathcal{M}$ is a definably connected ordered structure, then the following two assertions are equivalent

   (a) $\mathcal{M}$ is o-minimal;

   (b) every non-empty definable subset $D \subset M$ has a secure boundary.

   You can make use of the results proven in the notes on o-minimality covered in class. For some extra credit, show that the condition of being definably connected is not necessary.

8. Let $\mathbb{R}_\natural$ be a nontrivial ultrapower of the reals $\mathbb{R}$ and view $\mathbb{R}$ as a subfield via the diagonal embedding. Call an element $\alpha \in \mathbb{R}_\natural$ an *infinitesimal*, if $0 < |\alpha| < r$, for every positive $r \in \mathbb{R}$; call α *infinite*, if $1/\alpha$ is infinitesimal; and call α *finite* if it is not infinite. Give an example of a non-zero infinitesimal. Show that the finite elements form a ring R and the infinitesimals $\mathfrak{M}$ are an ideal in R. Show that for any element $\alpha \in \mathbb{R}_\natural$, either α is infinite or else there is a unique real number $r \in \mathbb{R}$, called the *standard part of* α, such that $\alpha - r$ is infinitesimal. Conclude that $R/\mathfrak{M} \cong \mathbb{R}$ and that $\mathfrak{M}$ is a maximal ideal.

Part Two (42 points)

Do THREE of the following six problems.

1. Fix the tape alphabet $\Sigma = \{0, 1, \llcorner\!\lrcorner\}$ for Turing Machines. Show that a set $A \subseteq \Sigma^*$ is recursively enumerable if and only if A can be reduced to the Halting Problem H. (You may also view A and H as subsets of ω, if you prefer.)

2. Work with standard Turing machines over finite alphabets. Say that a state of such a Turing machine is *unused* if the Turing machine never enters that state during its run on empty input. Consider the problem of determining whether a given Turing machine M has an unused state. Determine whether this problem is decidable, recursively enumerable (recognizable), or both, or neither. Prove that your answer is correct.

3. Show (in ZFC): if κ is an infinite cardinal, then $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.

   Hint: Given a sequence $\langle f_i \mid i < \kappa \rangle$ of functions from $\mathrm{cf}(\kappa)$ to κ, use a diagonal argument to find a function $g : \mathrm{cf}(\kappa) \to \kappa$ that's not listed.

4. Show (in ZF): if κ is an uncountable regular cardinal and $S \subseteq \kappa$ is stationary in κ, then there is a set $\bar{S} \subseteq S$, stationary in κ, such that whenever $\alpha \in \bar{S}$, then $\bar{S} \cap \alpha$ is not stationary in α. Here, we extend the definitions of club and stationary sets to subsets of arbitrary limit ordinals.

   Hint: First, note that we may assume that S consists only of limit ordinals. Now define whether $\alpha \in \bar{S}$ or not by recursion on α in the obvious way.

5. Suppose that Σ is a consistent, decidable, deductively closed set of sentences in a language with finitely many symbols. Prove or disprove that Σ can be extended to a consistent, complete, decidable set of sentences.

6. For a model M of some first order language, let $\mathsf{D}(M)$ be the set of $a \in M$ such that a is definable in M without parameters (that is, for some formula $\varphi(x)$, $M \models \varphi(a)$, but $M \models \neg\varphi(b)$ for all $b \in M$ with $b \neq a$). As is common, we will say that a set $A \subseteq M$ is *definable in M (without parameters)* if there is a formula $\varphi(x)$ such that $A = \{a \mid M \models \varphi(a)\}$.

   We will be interested in the case that $M \models \mathsf{ZFC}$ is a model of the language of set theory. So assume ZFC is consistent.

   (a) Show that if $M \prec N$ are models (such that M is an elementary substructure of N), then $\mathsf{D}(N) \subseteq \mathsf{D}(M)$.

   (b) Suppose $M \models \mathsf{ZFC}$ and $\mathsf{D}(M)$ is definable in M (without parameters). Show that $\mathrm{On}^M \subseteq \mathsf{D}(M)$, and conclude that $\mathsf{D}(M) = \mathsf{OD}^M$.

   (c) Use the first two points to show that there is a (countable) model M of ZFC such that $\mathsf{D}(M)$ is not definable in M (without using parameters).

   (d) Show that every model N of $\mathsf{ZFC} + V = \mathsf{HOD}$ has a countable elementary submodel M such that $M = \mathsf{D}(M)$, and in particular, $\mathsf{D}(M)$ is definable in M.