Department of Mathematics
The CUNY Graduate Center
Complex Analysis Qualifying Exam

August 2022

Instructions

The exam has three parts. Only the indicated number of questions will be counted to determine your score. *If you end up doing more, you must specify which problems you would like to be graded.* You have 3 hours to complete your work.

Notation

- $\mathbb{C}$: The complex plane
- $\mathrm{Re}(z), \mathrm{Im}(z)$: The real and imaginary parts of a complex number z
- $\Delta := \{z \in \mathbb{C} : |z| < 1\}$: The open unit disk
- $\mathcal{U} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$: The upper half-plane
- By a "region" we mean a non-empty connected open set in $\mathbb{C}$
- By a "conformal map" $\Omega \to \Omega'$ we mean a one-to-one holomorphic map of Ω onto Ω'

Part A. Answer any two of the following three questions.

A1. Suppose $\{u_n\}$ is a sequence of real-valued harmonic functions in a region U such that $u_1 \leq u_2 \leq u_3 \leq \cdots$, and set $u = \lim_{n\to\infty} u_n$. Show that either $u = +\infty$ everywhere in U, or u is harmonic in U and the convergence $u_n \to u$ is uniform on compact subsets of U.

A2. Define the Green's function of a region U with the singularity at $a \in U$. Let U_1 and U_2 be regions and $f : U_1 \to U_2$ be a conformal map. Let $a_1 \in U_1$ and $a_2 = f(a_1)$. If g_1 and g_2 are the Green's functions of U_1 and U_2 with the singularities at a_1 and a_2 respectively, prove that $g_1(z) = g_2(f(z))$ for all $z \in U_1$.

A3. Define the class $\mathcal{S}$ of schlicht functions in the unit disk Δ. What does Koebe's 1/4-theorem assert about schlicht functions? Give an example of a schlicht function which shows that the constant 1/4 in this theorem is optimal.

Part B. Solve any two of the following three problems.

B1. Let $f : \Delta \to \mathcal{U}$ be holomorphic, with $f(0) = i$. Prove the following:

(i) $\dfrac{1-|z|}{1+|z|} \leq |f(z)| \leq \dfrac{1+|z|}{1-|z|}$ for all $z \in \Delta$.

(ii) $|f'(0)| \leq 2$.

B2. Let U_1 and U_2 be simply connected regions neither of which is the whole plane. Suppose $f : U_1 \to U_2$ is a conformal map, $a_1 \in U_1$ and $a_2 = f(a_1)$. If $h : U_1 \to U_2$ is any holomorphic function with $h(a_1) = a_2$, show that $|h'(a_1)| \leq |f'(a_1)|$.

B3. Let U be a region. What can you say about a non-constant holomorphic function $f : U \to U$ which satisfies $f \circ f = f$ everywhere in U?

PART C. SOLVE ANY FOUR OF THE FOLLOWING SIX PROBLEMS.

C1. Let f and g be entire functions and $h = e^f + e^g$. Show that either h has no zeros or it has infinitely many zeros.

C2. Prove that for every real number $\lambda > 1$ the function $f(z) = ze^{\lambda - z} - 1$ has exactly one zero in Δ. Verify that this zero is real and positive.

C3. Let U be a bounded region. Suppose f and g are continuous and zero-free on the closure $\overline{U}$ and holomorphic in U. If $|f(z)| = |g(z)|$ for all $z \in \partial U$, show that there is a complex number λ with $|\lambda| = 1$ such that $f(z) = \lambda g(z)$ for all $z \in \overline{U}$.

C4. Suppose f is holomorphic in Δ and $f(0) \in \Delta \smallsetminus \{0\}$. If $\mathrm{Re}(f'(z)) > 0$ for all $z \in \Delta$, show that f has no fixed point along the segment $[0, f(0)]$. (Hint: For every $p \in [0, f(0)]$, $f(p) - p = f(0) + \int_{[0,p]} (f'(z) - 1)\, dz$.)

C5. Show that the sequence $\{f_n\}$ of entire functions defined by $f_1(z) = \sin(z)$ and $f_{n+1}(z) = \sin(f_n(z))$ for $n \geq 1$ is not normal in any neighborhood of the origin.

C6. Construct an entire function which has a zero of order 2 at every point of the sequence $\{\log n : n \geq 1\}$ and does not vanish anywhere else.