Qualifying Exam, *Real Analysis*
August 19th, 2022
2pm-5pm

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
TOTAL	60	

Instructions:

1. The exam will be 3 hours
2. This exam contains nine problems, but at most **SIX** problems will be graded. Please clearly list these here or on the first page.
3. Justify your answers. Where appropriate, state without proof the results you are using. Each part of a problem counts equally.

Problem 1

Let $(X, \mathcal{F}, \mu)$ be a measure space and f be a measurable function on X. Prove that the limit (possibly infinite)

$$\lim_{n\to\infty} \int_X |f|^{1/n} d\mu$$

exists and find it.

Problem 2

Consider the sequence

$$f_n(x) = e^{-nx} - 2e^{-2nx}, \quad x > 0$$

(a) Show that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ is well defined for each $x > 0$, and determine f.

(b) Show that the Lebesgue Theorem fails, namely

$$\sum_{n=1}^{\infty} \int_{(0,\infty)} f_n(x) \neq \int_{(0,\infty)} \sum_{n=1}^{\infty} f_n(x).$$

Problem 3

(a) For $f \in L^2(\mathbb{R})$ (with the Lebesgue measure) and a sequence of real numbers $\{x_n\}$ which converges to zero, define $f_n(x) = f(x + x_n)$. Show that f_n converges to f in $L^2(\mathbb{R})$. Hint: you may use the fact that L^2 functions can be approximated by continuous functions with compact support without a proof.

(b) Let A be a Lebesgue measurable set of positive Lebesgue measure: $\lambda(A) > 0$. Show that the set of differences $A - A := \{x - y \in \mathbb{R} \mid x, y \in A\}$ contains an open neighborhood of the origin.

Problem 4

Show that the function $f(x) = \frac{1}{2}\mathbb{1}_{(-1,1)}(x)$ can not be written as a convolution $f(x) = (g \circ g)(x)$ (a.e.) for any non-negative $g \in L^1(\mathbb{R})$. Hint: argue that $g(x) = 0$ a.e. outside of a compact set, use basic properties of Fourier transform and consider point where the Fourier transform of f is 0.

Problem 5
Denoting by B_r to be the open ball of $\mathbb{R}^N$ centered at the origin of radius r, consider a sequence $f_n \in L^2(B_1)$ which is bounded in the L^2-norm. Prove that f_n converges weakly to 0 in $L^2(B_1)$ if and only if the sequence

$$F_n : [0,1] \to \mathbb{R}, \quad r \mapsto \int_{B_r} f_n$$

converges uniformly to zero

Problem 6 (Application of Baire Theorem)
Let $\mathcal{P}$ be the vector space of all real polynomials in one variable. Prove that there is no norm which would turn $\mathcal{P}$ into a Banach space.

Problem 7

Consider the space $E := C_b^0(\mathbb{R}^n, \mathbb{C})$ consisting of continuous and bounded functions on $\mathbb{R}^n$ valued in $\mathbb{C}$ with the sup norm $\|f\|_\infty := \sup_{x\in\mathbb{R}^n} |f(x)|$. Given $a \in \mathbb{R}^n \setminus \{a\}$ define the continuous linear operator

$$T : E \to E, \qquad f \mapsto f(\cdot + a).$$

(a) What is the operator norm of T ?

(b) Show that the set of values $\lambda \in \mathbb{C}$ for which the operator $T - \lambda\mathrm{Id}$ is not injective is precisely

$$\{\lambda \in \mathbb{C} \, : \, |\lambda| = 1\}.$$

(c) Show that for $|\lambda| > 1$ the operator $T - \lambda\mathrm{Id}$ has an inverse which is given by $-\lambda^{-1} \sum_{k=0}^{\infty} (\frac{T}{\lambda})^k$.

(d) Show that T is invertible, and that for each $0 < |\lambda| < 1$ the operator $T - \lambda\mathrm{Id}$ has an inverse.

Problem 8
Let $\mathcal{H}$ be a Hilbert space and consider the set of bounded linear operators $\mathcal{L}(\mathcal{H}, \mathcal{H})$.

(a) Show there is a unique $T^* \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ such that $(Tx, y) = (x, T^*y)$ for all $x, y \in \mathcal{H}$. If $\mathcal{H} = \mathbb{R}^n$, what does T^* correspond to ?

(b) Show that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$.

(c) Let $\mathcal{R}(T)$ denote the range of T and $\mathcal{N}(T)$ denote the kernel/nullspace of T, i.e.

$$\mathcal{R}(T) = \{y : \exists x \in \mathcal{H} \ s.t. \ Tx = y\} \qquad \mathcal{N}(T) = \{x \in \mathcal{H} : Tx = 0\} \ .$$

Show that $\mathcal{R}(T)_\perp = \mathcal{N}(T^*)$ and $\mathcal{N}(T)_\perp = \overline{\mathcal{R}(T^*)}$.

Problem 9

(a) Let ν be a signed measure on $(X, \mathcal{M})$ such that $\nu = \mu_1 - \mu_2$ for two positive measures μ_1, μ_2. Show that $\mu_1(E) \geq \nu^+(E)$ and $\mu_2(E) \geq \nu^-(E)$ for any $E \in \mathcal{M}$.

(b) Use the previous question to show that for any two signed measures ν_1 and ν_2 with finite total variation, we have

$$\|\nu_1 + \nu_2\|_{TV} \leq \|\nu_1\|_{TV} + \|\nu_2\|_{TV} \ ,$$

where $\|\nu\|_{TV} = |\nu|(X) = \nu^+(X) + \nu^-(X)$.