Topology Qualifying Exam **August 2022**

Instructions: Do 8 problems in total, with exactly two problems from Part I, and at least two problems from each of Parts II and III. If you attempt more than 8 problems, identify which 8 should be graded. Justify your answers and clearly indicate which "well-known" theorems you cite.

Part I

1. Suppose that (X, τ) is a topological space and that τ is closed under arbitrary intersections. Prove that (X, τ) is Hausdorff if and only if (X, τ) is discrete.

2. Let A be a non-empty subset of the metric space X and define $f(x) = \inf\{d(x, a) \mid a \in A\}$. Show that $f(x) = 0$ if and only if $x \in \overline{A}$.

3. Let I be the unit interval $[0, 1]$ in $\mathbb{R}$ and let $X = \mathcal{C}(I, I)$ be the space of continuous maps from I to I with the compact-open topology. For each $x \in I$, let

$$U_x = \left\{ f : I \to I : |f(x) - x| < \frac{1}{2} \right\}.$$

(a) Prove that the collection $\{U_x\}_{x \in I}$ is an open cover of X.

(b) Prove or disprove: the collection $\{U_x\}_{x \in I}$ has a finite subcover.

4. Let X and Y be spaces and let $f : X \to Y$. Prove that f is a continuous injection if and only if the following diagram is a pullback square:

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Part II

5. Let $X = S^1 \vee S^1$ be the figure-8 graph with loops labeled a, b. Let $f \colon X \to X$ be a map such that $f_*(a) = ba$ and $f_*(b) = bab$. Let Y be the *mapping torus* of f:

$$Y = X \times [0, 1]/\sim, \text{ where } (x, 0) \sim (f(x), 1).$$

Construct a Δ-complex structure on Y, and use it to give a presentation of $\pi_1(Y)$.

6. Find three connected non-homeomorphic 2–fold covering spaces of $\mathbb{R}P^2 \vee S^1$.

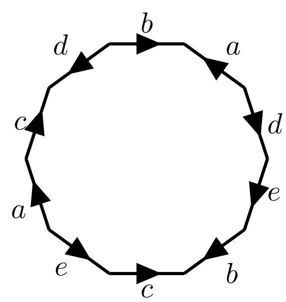
(a) Justify algebraically.

(b) Describe the covers using a sketch or otherwise.

7. Prove that if X is a path connected space and $x, y \in X$ then the based loop spaces $\Omega(X, x)$ and $\Omega(X, y)$ are homotopy equivalent.

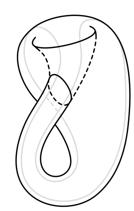
8. Let X be the quotient space of a cube I^3 obtained by identifying each pair of opposite square faces with a right-handed quarter-twist. Find a presentation for $\pi_1(X)$.

9. State the classification of closed surfaces. Compute the Euler characteristic of the surface obtained by identifying the sides of the polygon drawn below, write down a presentation for it's fundamental group, and identify which surface it is.



Part III

10. On the Klein bottle K, let γ be the small closed curve shown in the figure. Let M be a Möbius band. Let $X = K \cup M/\sim$, where γ is identified with ∂M. Use the Mayer-Vietoris theorem to compute the homology groups of X.



11. Let Top be the category of topological spaces and let Ab be the category of graded abelian groups.

(a) Describe singular homology as a functor $H : \mathsf{Top} \to \mathsf{Ab}$.

(b) Does the functor H have a left or right adjoint $\mathsf{Ab} \to \mathsf{Top}$?

12. Recall that $H^{2n}(\mathbb{CP}^n, \mathbb{Z}) \simeq \mathbb{Z}$. A map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ is *orientation preserving* if the map $f^* : H^{2n}(\mathbb{CP}^n) \to H^{2n}(\mathbb{CP}^n)$ is multiplication by a nonnegative integer. Prove that if n is even, then every map $f : \mathbb{CP}^n \to \mathbb{CP}^n$ is orientation preserving.

13. Let T and K denote the torus and Klein bottle. Prove that for any map $f \colon T \to K$, the map $f^* \colon H^2(K; \mathbb{Z}_2) \to H^2(T; \mathbb{Z}_2)$ is trivial. You may use the cup product structure on the cohomology of these spaces without proof as long as you state it clearly.

14. Let M be a closed, connected, orientable 4–manifold with $\pi_1(M) \cong \mathbb{Z} * \mathbb{Z}$ and $\chi(M) = 5$.

(a) Compute $H_i(M, \mathbb{Z})$ for all i.

(b) Let X be a CW-complex with no 3–cells. Show that M is not homotopy equivalent to X.