Differential Geometry Qualifying Exam
Fall 2022

Do any **6 problems**.
Note: Throughout this exam, all manifolds are C^∞ and connected, and all maps are C^∞ unless it is specifically stated otherwise.

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1. Consider $\mathbb{R}^2$ with the metric given by $g = e^{2f}(dx \otimes dx + dy \otimes dy)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.

   Prove that the Gaussian curvature of this manifold is $K = -e^{-2f}\Delta f$, where $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

   [Hint: use a local orthonormal coframe and the structure equations.]

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2. Consider a mapping φ from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $\varphi(x,y) = (x^2 + y^2, 1)$. Compute $\varphi^*(dx)$, $\varphi^*(dy)$, $\varphi^*(ydx)$ and $\varphi^*(xdy)$.

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3. Let M be a connected manifold and let U and V be open subsets of M such that $M = U \cup V$ and $U \cap V$ is connected. Prove that if ω is a 1-form on M such that $\omega_{|U}$ is exact and $\omega_{|V}$ is exact, then ω is itself exact.

   [Hint: Use the Mayer-Vietoris sequence for DeRham cohomology.]

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4. Consider the sphere S^2 as a submanifold of $\mathbb{R}^3$ in the usual way. Let $x : S^2 \to \mathbb{R}$ be defined by $x(p) = x$-coordinate of p. Let $y : S^2 \to \mathbb{R}$ and $z : S^2 \to \mathbb{R}$ be defined similarly.

   (a) Show that $dz = -\dfrac{xdx + ydy}{z}$, wherever z 6= 0.

   (b) Calculate the integral $\int_D dx \wedge dz + dy \wedge dz$, where D is the upper hemisphere (that is, the part of the sphere between the equator $z = 0$ and the north pole).

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5. Let $M = \mathbb{R}^2$ with the usual coordinates (x,y), and let $X = 2\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$.

   (a) Find explicitly the integral curve $\gamma_{(x,y)}(t)$ of X such that $\gamma_{(x,y)}(0) = (x,y)$.

   (b) If $\theta_t(x,y) := \gamma_{(x,y)}(t)$ is the flow of X, find the matrix of the derivative $(\theta_t)_{*(x,y)} : T_{(x,y)}M \to T_{\theta_t(x,y)}M$ in the standard basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$.

$Y = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ (c) If, find $(\theta_{-t})_{*\,(x,y)}(Y_{\theta_t(x,y)}) \in T_{(x,y)}M$.

(d) Find $L_X(Y)$ (the Lie derivative) and verify that it is equal to $[X,Y]$.

6. Show that if a Riemannian manifold M has the property that for every two points $p,q \in M$ and orthonormal pairs of tangent vectors $v,w \in T_pM$ and $\tilde{v},\tilde{w} \in T_qM$ there exists a local isometry taking p to q, v to $\tilde{v}$, and w to $\tilde{w}$, then M has a constant sectional curvature.

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7. Consider the metric in $\mathbb{R}^2$ given by $g = dx \otimes dx + e^{2y} dy \otimes dy$.

(a) Find the Gaussian curvature of this metric. [Hint: use a moving frame.] (b) Show that the curve x = constant is a geodesic.

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8. Consider the Lie group

$$SU(2) = \{A \in GL(2,\mathbb{C}) : A\overline{A^t} = I_2\}$$

(a) Show that $SU(2)$ is diffeomorphic to S^3.

(b) Show that the tangent bundle to S^3 is trivial.

(c) Consider the map

$$\pi : S^3 \to \mathbb{C} \times \mathbb{R}, (a,z) \mapsto (2a\bar{z}, |a|^2 - |z|^2)$$

Show that the map π is smooth and that its range is S^2. Given $x \in S^2$, characterize the set $\pi^{-1}(x)$.

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9. Given $m,n \in \mathbb{N}$, consider the set

$$\Sigma(m,n) = \{[z_0 : z_1 : \cdots : z_m] \in \mathbb{C}P^m : \sum_{k=0}^{m} z_k^n = 0\}$$

Show that this a submanifold of $\mathbb{C}P^m$. What is its dimension?

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10. Draw a surface of dimension 2 of genus g, and indicate a point where the curvature is positive, a point where the curvature is negative, and a point where the curvature is zero.

Prove that if M is a 2-dimensional Riemannian manifold of genus $g > 1$, then there exists an open subset of M where the curvature is negative.

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11. Let M be a submanifold of $\mathbb{R}^n$, with $n \geq 2$, of codimension two, such that $0 \notin M$.

    Show that there exists a one-dimensional subspace $V \in \mathbb{RP}^{n-1}$ such that $V \cap M = \emptyset$.