CUNY GRADUATE CENTER
DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAM

Fall 2022
3 hours

Instructions. The exam consists of two parts. Choose a total of six problems, including two from each part. Indicate on the front cover of the exam booklet the problems you have chosen. Partial credit will be awarded generously, but only for those problems. Justify your answers. State clearly any major theorems that you are using to obtain your results.

Part 1.

1. Show that if n is odd then the set of all n-cycles consists of two conjugacy classes of equal size in A_n.

2. Let G be a finite group and let $H \leq G$ be a subgroup of index $|G : H| = n$.
(a) Show that $|H : (H \cap H^g)| \leq n$ for all $g \in G$.
(b) If H is a maximal proper subgroup of G and H is abelian, show that $(H \cap H^g)$ is a normal subgroup of G for all $g \notin H$.
(c) Now suppose that G is simple. If H is abelian and n is a prime, prove that $H = 1$.

3. How many non-isomorphic groups are there of order pq, where p, q are primes? Describe them.

4. Describe all isomorphism classes of abelian groups of order 100 (use invariant factors or elementary divisors).

5. Let R be a ring where every left ideal is a projective left R-module. Prove that every submodule of a finitely generated free R-module is projective. (Hint: induct on the number of generators.)

6. Prove that if R is a UFD, then there are no infinite strictly increasing sequences$(a_1) \subsetneq (a_2) \subsetneq \cdots$ of principal ideals.

Part 2.

7. Let R be a commutative ring.
(a) Prove that $R/(a) \otimes_R M \cong M/aM$ for any left R-module M. (Check $r + (a) \otimes m \to rm + aM$ is a well defined homomorphism with inverse $m + aM \mapsto 1 + (a) \otimes m$.)
(b) Let $a, b \in R$. Prove that $R/(a) \otimes_R R/(b) \cong R/(a, b)$.

8. Find the Rational Canonical Form and the Jordan Form of the matrices

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}$$

. Are these matrices similar?

9. Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$.

10. Prove there are only a finite number of roots of unity in any finite extension K of $\mathbb{Q}$.

11. Determine the minimal polynomial over $\mathbb{Q}$ of the element $\sqrt{2} + \sqrt{3}$. Determine the Galois group of this polynomial.

12. What is the radical of the principal ideal $(u^3 - uv^2 + v^3)$ in $\mathbb{Q}(u, v)$?